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Time Causal Processes in Time Petri Nets with Weak Semantics

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Abstract. In this paper, we present a method for state space reduction of dense-time Petri nets (TPNs) – an extension of Petri nets by adding a time interval to every transition for its firing. The time elapsing and memory operating policies define different semantics for TPNs. The decidability of many standard problems in the context of TPNs depends on the choice of their semantics. The state space of the TPN is infinite and non-discrete, in general, and, therefore, the analysis of its behavior is rather complicated. To cope with the problem, we elaborate a state space discretization technique and develop a partial order semantics for TPNs equipped with weak time elapsing and intermediate memory policies.

Keywords: dense-time Petri nets; weak time elapsing semantics; intermediate memory policy; state space discretization; partial order semantics; time causal processes

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Временные причинно-упорядоченные процессы временных сетей Петри со «слабой» семантикой

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Abstract. В данной статье предлагается метод редукции пространства состояний непрерывно-временных сетей Петри (НВСП) – расширения сетей Петри, где каждому переходу ставится в соответствие временной интервал его срабатывания. Техники контроля времени и памяти определяют различные семантики для НВСП, которые влияют на разрешимость многих стандартных проблем анализа поведения НВСП. В общем случае, пространство состояний НВСП бесконечно и несчетно, и, следовательно, анализ их поведения довольно сложен. С целью разрешения данной проблемы выполняется дискретизация пространства состояний и определяется семантика частичного порядка для НВСП со «слабой» техникой продвижения времени (продвижение времени неограничено) и «промежуточной» техникой контроля памяти (с учетом промежуточных разметок при срабатывании сетевых переходов).

Ключевые слова: непрерывно-временные сети Петри; «слабая» семантика продвижения времени; «промежуточная» техника сброса часов; дискретизация пространства состояний; семантика частичного порядка, временные причинные процессы

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1 Introduction

Dense-Time Petri Nets (TPNs) are now a well-established model to describe and study safety-critical systems that often require verification of real time (quantitative) characteristics, in addition to functional (qualitative) properties. In the TPN, each transition is associated with a time interval. With that, each transition is assumed to have its own local clock. A state of the TPN contains a current marking and readings of the local clocks of enabled transitions (i.e. transitions whose all input places have enough tokens at the marking). A transition can fire from a state only if the transition is enabled at the corresponding marking and its clock reaches a moment in time that is within the interval associated. So, the firing of an enabled transition can be suspended for a certain time. Along with that, the firing itself takes no time. State changes are divided in two types: either *time elapses*, i.e. the clocks of enabled transitions go forward, or *a transition fires*, i.e. a current marking is changed to a new one (in which tokens are consumed from the input places and tokens are produced to the output places of the transition that fires) and the clocks of the transitions that become enabled at the new marking (newly enabled transitions) are reset to zero.

There are two policies of time elapsing in TPNs, which define strong and weak semantics. In the former semantics, time elapsing cannot exceed the upper bounds of enabled transitions and, therefore, an enabled transition must fire no later than the upper bound of its time interval is reached. On the contrary, any time elapsing is allowed in the latter semantics and, therefore, enabled transitions are not forced to fire. In [1], the authors have proven that the two semantics are incomparable w.r.t. timed weak bisimulation.

Memory policies in TPNs determine when the local clocks of enabled transitions are reset. Intermediate and atomic memory policies are put forward in the literature. The former treats intermediary marking, i.e. the marking after consumption of tokens from the input places and before production of tokens to the output places of a transition t that fires. A transition t' is regarded as newly enabled and its clock is reset to zero after the firing of t whenever t' is disabled at the intermediary marking and becomes enabled at the new marking, i.e. after production of tokens to the output places of t. Instead, the latter policy considers a firing as one-step. The clock of t' is reset to zero only if it is disabled at the marking before t fires and becomes enabled at the new marking after t fires. The memory policies were studied in [2] for strong semantic and in [3] for weak semantics. It was shown that the marking reachability/coverability and boundedness problems are undecidable for time Petri nets with strong semantics and any memory policy, whereas the problems are decidable in the case of TPNs with weak intermediate semantics but not with weak atomic semantics.

The state space of the TPN is infinite and non-discrete, in general, that increases the difficulty of the model analysis. In the work [4], a transformation to the behavior with only integer time elapsings has been suggested for TPNs with strong semantics, while the discretization of the state space for weak semantics has hitherto not be treated in the literature, to the best of our knowledge.

The classical interleaving behavior of the TPN is described by runs – sequences of changes in states by time elapsings or transition firings. Interleaving semantics allows for analyzing some safety and liveness properties of systems. However, using partial order semantics seems preferable because it captures in a natural way «true concurrency». Partial order semantics of Petri nets is most often represented by means of the so-called causal net processes, which include events and conditions related by causal dependence and concurrency. This information can be useful for formal verification of the system behavior or for reducing the number of analyzed system states, without taking into account all interleaving sequences. Partial order semantics is put forward for safe TPNs with strong and clocks-on-transitions semantics in [5]. The presented in [6] approach to construct a partial order and non-deterministic representation of the behavior of safe TPNs with strong and clocks-on-tokens semantics consists in transforming time characteristics into net structure, i.e. representing them by additional places, transitions, and arcs. This allows for removing the restrictions of diverging time and of finite upper time bounds for transitions. In [7], the authors inspect free choice TPNs (i.e. net transitions sharing an input place do have exactly the same input places), develop and compare partial order representations of runs, based on various clocks-on-tokens semantics.

In this paper, we deal with dense-time Petri nets with weak and intermediate policies. Our intention here is twofold. First, we develop a discrete representation of the interleaving behavior (runs) of the TPN by transforming its runs with real-number time elapsings to parametric sequences with time variables that are then assigned natural-number values. Second, partial order clocks-on-tokens semantics in terms of time causal processes of the TPN, by converting time elapsings into net structure, is elaborated. Also, for the TPN, a bijective mapping between its runs and computations (called linearizations) of its time causal processes is constructed, in order to demonstrate the correctness of the partial order semantics w.r.t. interleaving one. Partial order semantics allow for taking into account the processes' timing behavior in addition to their degrees of relative concurrency.

The paper is organized as follows. In Section 2, we consider some definitions for TPNs and their interleaving semantics in terms of runs – sequences of changes in states by time elapsings and transition firings. In the following section, it is established that the discretization of TPN's state space is possible by demonstrating that in the TPN for any run with transition firings and real-number time elapsings there exists a run having the same transition firings and only natural-number (even unit) time elapsings. In Section 4, we introduce and examine properties of a casual net, its linearization, and a time causal process of the TPN, consisting of a casual net and its homomorphism into the TPN. In the next section, a bijective mapping from a set of linearizations of causal nets of time processes of the TPN to its set of runs is developed and studied. Section 6 concludes the paper.

2 Time Petri Nets

In this section, some terminology concerning the model of Petri nets with timing constraints (time intervals on the firings of transitions) are defined. We start with recalling the definitions of the structure and behavior of Petri nets.

The Petri net (PN) consists of two different sets of elements – places and transitions; a flow relation representing arrows between the elements; an initial marking – a subset of places initially containing tokens; and a labeling function mapping each transition to an action from the alphabet Act of actions. A state of the PN is called a marking – a subset of places that receive tokens when the net functions. A transition is enabled at a marking if the input places of the transition contain tokens. The firing of a transition enabled at a marking results in the new marking in which tokens are consumed from the input places and tokens are produced to the output places of the transition. A sequence of changes in markings is called a run of the PN.

Definition 1. A (*labeled over* Act) *Petri net* (PN) is a tuple $\mathcal{N} = (P, T, F, M_0, L)$, where P is a finite *set of places* and T is a finite *set of transitions* such that $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$; $F \subseteq (P \times T) \cup (T \times P)$ is a *flow relation*; $\emptyset \neq M_0 \subseteq P$ is an *initial marking*; $L : T \to Act$ is a *labeling function*. For $x \in P \cup T$, let $\bullet x = \{y | (y, x) \in F\}$ and $x\bullet = \{y | (x, y) \in F\}$ be the *preset* and *postset* of x, respectively. For $X \subseteq P \cup T$, define $\bullet X = \bigcup_{x \in X} \bullet x$ and $X\bullet = \bigcup_{x \in X} x\bullet$.

A *marking* M of a Petri net $\mathcal{N}$ is any subset of P. A transition $t \in T$ is *enabled at a marking* M if $\bullet t \subseteq M$. Let $En(M)$ be the set of transitions enabled at M.

The *firing of a transition* t enabled at a marking M leads to the new marking M' (denoted $M \xrightarrow{t} M'$) iff $M' = (M \setminus \bullet t) \cup t\bullet$. We write $M \xrightarrow{\vartheta} M'$ iff $\vartheta = t_1 \ldots t_k$ and $M = M^0 \xrightarrow{t_1} M^1 \ldots M^{k-1} \xrightarrow{t_k} M^k = M'$ $(k \geq 0)$. In this case, ϑ is a *run* of $\mathcal{N}$ from M (to M'), and M' is a *reachable marking of* $\mathcal{N}$ *from* M. Let $\mathcal{RM}(\mathcal{N})$ be the set of all reachable markings of $\mathcal{N}$ from M_0.

The time Petri net (TPN) consists of an underlying PN and a static timing function mapping each transition to a time interval with non-negative rational boundaries. With that, each transition is assumed to have its own local clock. A marking alone is not enough to describe a state of the TPN, so a dynamic timing function is added to indicate the clock values of the transition enabled at a current marking. In fact, the clocks of enabled transitions show the times passed since then as the transitions become enabled. The initial state consists of the initial marking and the dynamic timing function with zero clock values for all enabled transitions. When the TPN is running, there are two ways to change states: either by time elapsings or by transition firings. Following the approach of [3], we consider TPNs with weak semantics. This means that any time elapsing is allowed, i.e. any time can be added to the clock values of enabled transitions. A transition can fire from a current state only if the transition is enabled at the current marking and its clock value belongs to its time interval. The firing of a transition that can fire from a state results in a new state, i.e. a new marking and new dynamic timing function with the clock values reset to zero for the newly enabled transitions and with the old clock values for the transitions which continue to be enabled. We deal with TPNs with intermediate memory policy, i.e. the predicate $\uparrow enabled(t', M, t)$ determining a newly enabled transition t' after the firing of a transition t at a marking M has a true value if and only if t' is disabled at intermediary marking (i.e. the marking between consumption and production of tokens by the firing of *t*) and becomes enabled at the new marking (i.e. the marking after production of tokens by the firing of *t*). A sequence of changes in states is called a run of the TPN. The runs from the initial state represent interleaving semantics of the TPN.

Definition 2. A (*labeled over* Act) *time Petri net* (TPN) is a pair $\mathcal{TN} = (\mathcal{N}, D)$, where $\mathcal{N} = (P, T, F, M_0, L)$ is the underlying Petri net and $D: T \to \mathbb{Q}_{\geq 0} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$ is a *static timing function* mapping each transition to a closed non-empty interval with non-negative rational boundaries; right open infinite boundaries are allowed. For a transition $t \in T$, the boundaries of the interval $D(t)$ are called the *earliest firing time* (Eft) and *latest firing time* (Lft) *of* t.

A *state of* $\mathcal{TN}$ is a pair $S = (M, I)$, where M is a marking and $I: En(M) \to \mathbb{R}_{\geq 0}$ is a *dynamic timing function*. The *initial state* of $\mathcal{TN}$ is a pair $S_0 = (M_0, I_0)$, where M_0 is the initial marking and $I_0(t) = 0$, for all $t \in En(M_0)$.

A transition t *can fire from a state* $S = (M, I)$ if $t \in En(M)$ and $Eft(t) \leq I(t) \leq Lft(t)$.

In the TPN, two types of state changes are possible by:

a) the *elapsing of time* $\tau \in \mathbb{R}_{\geq 0}$, defined as follows:

$$(M, I) \xrightarrow{\tau} (M, I') \text{ iff } \forall t' \in En(M) : I'(t') = I(t') + \tau;$$

b) the *firing of a transition* $t \in T$, defined as follows:

$$(M, I) \xrightarrow{t} (M', I') \text{ iff } \begin{cases} t \text{ can fire from } (M, I), \text{ and} \\ M' = (M \setminus \bullet t) \cup t\bullet, \text{ and} \\ \forall t' \in En(M') : I'(t') = \begin{cases} 0, & \text{if } \uparrow enabled(t', M, t), \\ I(t'), & \text{otherwise,} \end{cases} \end{cases}$$

where the predicate

$$\uparrow enabled(t', M, t) = t' \in En((M \setminus \bullet t) \cup t\bullet) \wedge (t' \notin En(M \setminus \bullet t) \vee t = t')$$

indicates whether we need to reset the clock of t' after the firing of t at M.

We use the notation $S \overset{\sigma}{\rightarrow} S'$ iff $\sigma = \overline{t_1} \dots \overline{t_k} \in (T \cup \mathbb{R}_{\geq 0})^k$ and $S = S^0 \overset{\overline{t_1}}{\rightarrow} S^1 \dots S^{k-1} \overset{\overline{t_k}}{\rightarrow} S^k = S'$ $(k \geq 0)$. In this case, σ is a *run* of $\mathcal{TN}$ *from* S (*to* S'), and S' is a *reachable state of* $\mathcal{TN}$ *from* S. Let $\mathcal{FS}(\mathcal{TN})$ be the set of all runs of $\mathcal{TN}$ from S_0, $\mathcal{RS}(\mathcal{TN})$ be the set of all reachable states of $\mathcal{TN}$ from S_0. We write $Untimed(\sigma)$ to denote the projection σ on T, i.e. the untimed part of σ.

Consider some properties of TPNs. We call $\mathcal{TN}$ *safe*, iff $M(p) \leq 1$, for all $S = (M, I) \in RS(\mathcal{TN})$ and $p \in P$; *contact-free* iff whenever t can fire from a state $S = (M, I)$, then $(M \setminus \bullet t) \cap t\bullet = \emptyset$ for all $S \in \mathcal{RS}(\mathcal{TN})$; T-*restricted* iff $\bullet t \neq \emptyset$ and $t\bullet \neq \emptyset$ for all transitions in the underlying Petri net.

Notice that the definition of the marking of the underlying PN as a subset, rather than a multiset, of the net places (see Definition 1) ensures that each place has at most one token when the TPN is functioning, i.e. it is safe. This leads to the fact that any transition can be enabled at most once at any marking M and can fire at most once from a corresponding state (M, I). As a consequence, $En(M)$ is a set, rather than a multiset, of transitions, and the dynamic timing function I is really a function, rather than a relation. The contact-freeness property says that a transition cannot fire from a state, if at least one output place (which is not the input place) of the transition already contains a token at the corresponding marking. In the case when the TPN is not contact-free, after the firing of an enabled transition from a state, two or more tokens can accumulate in the output places of the transition. However, some of the tokens may be lost, as the marking is defined as a subset, rather than a multiset, of places. Due to the *T*-restrictedness property, each net transition has at least one input place and at least one output place. This allows us to avoid livelock (useless work) situations as the transitions without input and output places can fire (work) infinitely many times without consuming and producing any tokens (results). So, the above properties facilitate the correct definitions and results concerning TPNs. In what follows, we will consider only safe, contact-free and *T*-restricted TPNs.

Example 1. A (labeled over $Act = \{a, b\}$) time Petri net $\widetilde{\mathcal{TN}}$ is shown in Fig. 1. Here, the places are represented by circles and transitions by squares; the names are depicted near the elements. The elements included in the flow relation are connected by arrows, and each place contained in the initial marking is that with a token (bold point). The values of the labeling and static timing functions are printed next to the transitions. It is easy to see that $\widetilde{\mathcal{TN}}$ is really safe, contact-free and T-restricted. Show that $\sigma = t_1 t_3 (2.3) t_2 (1.5) t_3$ is a run of $\mathcal{TN}$ from S_0.

- $S_0 = (M_0, I_0)$, where $M_0 = \{p_1, p_2\}, En(M_0) = \{t_1, t_3\}$ and $\forall t \in En(M_0): I_0(t) = 0$.
- Due to $t_1 \in En(M_0)$ and $Eft(t_1) = 0 \leq I_0(t_1) = 0 \leq Lft(t_1) = 1$, we have that t_1 can fire from (M_0, I_0). Then, $S_0 \overset{t_1}{\rightarrow} S_1 = (M_1, I_1)$, where $M_1 = (M_0 \setminus \bullet t_1) \cup t_1 \bullet = \{p_2, p_3\}$, $En(M_1) = \{t_3\}$ and $I_1(t_3) = I_0(t_3) = 0$, because $\uparrow enabled(t_3, M_0, t_1) = false$.
- Due to $t_3 \in En(M_1)$ and $Eft(t_3) = 0 \leq I_1(t_3) = 0 \leq Lft(t_3) = 2$, we have that t_3 can fire from (M_1, I_1). Then, $S_0 \overset{t_1 t_3}{\longrightarrow} S_2 = (M_2, I_2)$, where $M_2 = (M_1 \setminus \bullet t_3) \cup t_3 \bullet = \{p_3, p_4\}$, $En(M_2) = \{t_2\}$ and $I_2(t_2) = 0$, because $\uparrow enabled(t_2, M_1, t_3) = true$.
- $S_0 \xrightarrow{t_1 t_3 (2.3)} S_3 = (M_3, I_3)$, where $M_3 = M_2 = \{p_3, p_4\}$, $En(M_3) = \{t_2\}$ and $I_3(t_2) = I_2(t_3) + 2.3 = 2.3$.
- Due to $t_2 \in En(M_3)$ and $Eft(t_2) = 1 \leq I_3(t_2) = 2.3 \leq Lft(t_2) = 3$, we have that t_2 can fire from (M_3, I_3). Then, $S_0 \xrightarrow{t_1 t_3 (2.3) t_2} S_4 = (M_4, I_4)$, where $M_4 = (M_3 \setminus \bullet t_2) \cup t_2 \bullet = \{p_1, p_2\}$, $En(M_4) = \{t_1, t_3\}$ and $I_4(t_1) = I_4(t_3) = 0$, because $\uparrow enabled(t_1, M_3, t_2) = \uparrow enabled(t_3, M_3, t_2) = true$.
- $S_0 \xrightarrow{t_1 t_3 (2.3) t_2 (1.5)} S_5 = (M_5, I_5)$, where $M_5 = M_4 = \{p_1, p_2\}$, $En(M_5) = \{t_1, t_3\}$ and $I_5(t_1) = I_4(t_1) + 1.5 = 1.5$, $I_5(t_3) = I_4(t_3) + 1.5 = 1.5$.
- Due to $t_3 \in En(M_5)$ and $Eft(t_3) = 0 \leq I_5(t_3) = 1.5 \leq Lft(t_3) = 2$, we have that t_3 can fire from (M_5, I_5). Then, $S_0 \xrightarrow{t_1 t_3 (2.3) t_2 (1.5) t_3} S_6 = (M_6, I_6)$, where $M_6 = (M_5 \setminus \bullet t_3) \cup t_3 \bullet = \{p_1, p_4\}$, $En(M_6) = \{t_1\}$ and $I_6(t_1) = I_5(t_1) = 1.5$, because $\uparrow enabled(t_1, M_5, t_3) = false$.

Therefore, $\sigma = t_1 t_3 (2.3) t_2 (1.5) t_3$ is a run of $\mathcal{TN}$ from S_0. □

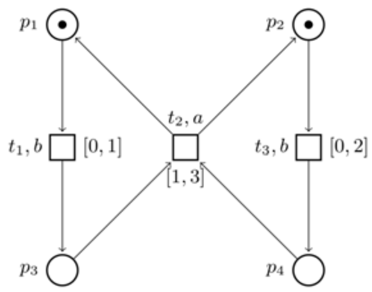

Fig. 1. *A time Petri net* $\widetilde{\mathcal{TN}} = (\widetilde{\mathcal{N}}, D)$

In order to display that every TPN can be transform into that with natural-valued boundaries of the intervals associated with its transitions, we need a notion of time equivalence. Two TPNs are considered time equivalent if they have the same underlying Petri net, and for each transition, its earliest and latest firing times in the TPNs are either proportional to a non-zero constant or its latest firing times are together equal to infinity.

Definition 3. Two time Petri nets $\mathcal{TN}_1 = (\mathcal{N}, D_1)$ and $\mathcal{TN}_2 = (\mathcal{N}, D_2)$ are *time equivalent* iff there exists a non-negative constant $c \neq 0$ such that for any transition t in $\mathcal{N}$ it holds:

- $Eft_2(t) = Eft_1(t) \cdot c$,
- $Lft_2(t) = \begin{cases} \infty, & \text{if } Lft_1(t) = \infty, \\ Lft_1(t) \cdot c, & \text{otherwise.} \end{cases}$

We next establish that for any TPN there is a time equivalent TPN with time intervals having natural-valued boundaries.

Theorem 1. Given a TPN $\mathcal{TN}_1 = (\mathcal{N} = (P, T, F, M_0, L), D_1)$, there exists a TPN $\mathcal{TN}_2 = (\mathcal{N}, D_2)$, with $D_2: T \rightarrow \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, such that $\mathcal{TN}_1$ and $\mathcal{TN}_2$ are time equivalent. Moreover, for any $\sigma_1 \in \mathcal{FS}(\mathcal{TN}_1)$, there is $\sigma_2 \in \mathcal{FS}(\mathcal{TN}_2)$ with the same transition firings and time elapsings multiplied by a constant c, and vice versa.

Proof. Construct the set $\mathcal{D}$ of the denominators of the boundaries from D_1 as follows: $\mathcal{D} = \left\{n \,\middle|\, Eft_1(t) = \frac{m}{n};\ t \in T; m, n \in \mathbb{N}_{>0}\right\} \cup \left\{n \,\middle|\, Lft_1(t) = \frac{m}{n};\ t \in T;\ m, n \in \mathbb{N}_{>0}\right\}$. Calculate the least common multiple of the denominators: $c = \begin{cases} 1, & \text{if } \mathcal{D} = \emptyset \\ LCM(\mathcal{D}), & \text{otherwise} \end{cases}$. Due to c being the least common multiple, we have non-negative constant $c \neq 0$. Define $D_2: T \rightarrow \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ as follows:

- $Eft_2(t) = Eft_1(t) \cdot c$,
- $Lft_2(t) = \begin{cases} \infty, & \text{if } Lft_1(t) = \infty, \\ Lft_1(t) \cdot c, & \text{otherwise}, \end{cases}$
- $D_2(t) = (Eft_2(t), Lft_2(t))$.

So, $\mathcal{TN}_1$ and $\mathcal{TN}_2$ are time equivalent.

Take an arbitrary $\sigma_1 = \overline{t_1} \dots \overline{t_k} \in \mathcal{FS}(\mathcal{TN}_1)$, with $(M_0, I_0) \xrightarrow{\overline{t_1}} (M_1, I_1) \dots (M_{k-1}, I_{k-1}) \xrightarrow{\overline{t_k}} (M_k, I_k)$. Construct $\sigma_2 = \overline{t_1'} \dots \overline{t_k'}$ such that $\overline{t_\iota} \in T \Longrightarrow \overline{t_\iota'} = \overline{t_\iota}$ and $\overline{t_\iota} \in \mathbb{R}_{\geq 0} \Longrightarrow \overline{t_\iota'} = \overline{t_\iota} \cdot c$, for all $1 \leq i \leq k$.

We shall prove that $\sigma_2 \in \mathcal{FS}(\mathcal{TN}_2)$, with $(M_0', I_0') \xrightarrow{\overline{t_1'}} (M_1', I_1') \dots (M_{k-1}', I_{k-1}') \xrightarrow{\overline{t_k'}} (M_k', I_k')$, by induction on $1 \leq i \leq k$.

$i = 0$. Then, $M_0' = M_0$, due to $\mathcal{TN}_1$ and $\mathcal{TN}_2$ having the same underlying Petri net; and $I_0'(t) = 0$, for all $t \in En(M_0')$, due to Definition 2.

$i > 0$. By the induction hypothesis, we have that $(M_0', I_0') \xrightarrow{\overline{t_1'} \dots \overline{t_{i-1}'}} (M_{i-1}', I_{i-1}')$. Thanks to the construction of σ_2, we obtain that $M_j' = M_j$, and, hence, $En(M_j') = En(M_j)$, for all $0 \leq j \leq i-1$, due to Definition 2. Then, it holds that $\uparrow enabled(t, M_{j-1}', \overline{t_j'}) = \uparrow enabled(t, M_{j-1}, \overline{t_j})$, for all $t \in En(M_{j-1}') = En(M_{j-1})$ and for all $1 \leq j \leq i$, i.e. the prefixes of σ_1 and σ_2 have the same clock resets for enabled transitions. This implies that $I_j'(t) = I_j(t) \cdot c$, for all $t \in En(M_j')$ and $0 \leq j \leq i-1$, due to Definition 2 and the construction of σ_2. Show that $(M_{i-1}', I_{i-1}') \xrightarrow{\overline{t_\iota'}} (M_i', I_i')$. Two cases are admissible.

1. $\overline{t_\iota'} \in \mathbb{R}_{\geq 0}$. Then, $\overline{t_\iota'} = \overline{t_\iota} \cdot c$, by the construction of σ_2. As $\sigma_1 \in \mathcal{FS}(\mathcal{TN}_1)$, we have that $M_i = M_{i-1}$ and $\forall t \in En(M_{i-1}) : I_i(t) = I_{i-1}(t) + \overline{t_\iota}$, due to Definition 2. Then, it holds that $I_i(t) \cdot c = I_{i-1}(t) \cdot c + \overline{t_\iota} \cdot c = I_{i-1}'(t) + \overline{t_\iota'} = I_i'(t)$, for all $t \in En(M_{i-1}' = M_i')$, Therefore, $(M_{i-1}', I_{i-1}') \xrightarrow{\overline{t_\iota'}} (M_i', I_i')$, according to Definition 2.
2. $\overline{t_\iota'} \in T$. Then, $\overline{t_\iota'} = \overline{t_\iota}$, by the construction of σ_2. As $\sigma_1 \in \mathcal{FS}(\mathcal{TN}_1)$, we have that $\overline{t_\iota} \in En(M_{i-1})$ and $Eft_1(\overline{t_\iota}) \leq I_{i-1}(\overline{t_\iota}) \leq Lft_1(\overline{t_\iota})$, because $\overline{t_\iota}$ can fire from (M_{i-1}, I_{i-1}). Hence, we obtain that $\overline{t_\iota'} \in En(M_{i-1}')$ and $Eft_2(\overline{t_\iota'}) = Eft_1(\overline{t_\iota}) \cdot c \leq I_{i-1}(\overline{t_\iota}) \cdot c = I_{i-1}'(\overline{t_\iota'}) = I_{i-1}(\overline{t_\iota}) \cdot c \leq \begin{Bmatrix} \infty, & \text{if } Lft_1(\overline{t_\iota}) = \infty, \\ Lft_1(\overline{t_\iota}) \cdot c, & \text{otherwise.} \end{Bmatrix} = Lft_2(\overline{t_\iota'})$, due to $\mathcal{TN}_1$ and $\mathcal{TN}_2$ being time equivalent with a proportionality constant c. So, $\overline{t_\iota'}$ can fire from (M_{i-1}', I_{i-1}'). Since $\sigma_1 \in \mathcal{FS}(\mathcal{TN}_1)$, we have that $M_i = (M_{i-1} \setminus \bullet\, \overline{t_\iota}) \cup \overline{t_\iota}\bullet$ and $\forall t \in En(M_i) : I_i(t) = \begin{cases} 0, & \text{if } \uparrow enabled(t, M_{i-1}, \overline{t_\iota}), \\ I_{i-1}(t), & \text{otherwise} \end{cases}$, by Definition 2. Then, it holds that $(M_{i-1}' \setminus \bullet\, \overline{t_\iota'}) \cup \overline{t_\iota'}\bullet = M_i' = M_i$ and $\forall t \in En(M_i') : I_i(t) \cdot c = \begin{cases} 0, & \text{if } \uparrow enabled(t, M_{i-1}', \overline{t_\iota'}), \\ I_{i-1}(t) \cdot c = I_{i-1}'(t), & \text{otherwise} \end{cases} = I_i'(t)$. Therefore, $(M_{i-1}', I_{i-1}') \xrightarrow{\overline{t_\iota'}} (M_i', I_i')$, thanks to Definition 2.

Thus, $(M_0', I_0') \xrightarrow{\sigma_2} (M_k', I_k')$, i.e. $\sigma_2 \in \mathcal{FS}(\mathcal{TN}_2)$.
Reasoning analogously, we can show that for each $\sigma_2 \in \mathcal{FS}(\mathcal{TN}_2)$, there exists $\sigma_1 \in \mathcal{FS}(\mathcal{TN}_1)$ such that σ_1 and σ_2 have the same untimed part. ◻

Example 2. Consider the PN $\widetilde{\mathcal{N}}$ whose structure is shown in Fig. 1. Define the TPN $\widetilde{\mathcal{TN}_1} = (\widetilde{\mathcal{N}}, D_1)$, with $D_1(t_1) = [0, \frac{1}{4}]$, $D_1(t_2) = [\frac{1}{4}, \frac{3}{4}]$, $D_1(t_3) = [0, \frac{1}{2}]$. Construct a TPN $\widetilde{\mathcal{TN}_2} = (\widetilde{\mathcal{N}}, D_2)$, with $D_2 : T \to \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ such that $\widetilde{\mathcal{TN}_1}$ and $\widetilde{\mathcal{TN}_2}$ are time equivalent. Define the set $\mathcal{D}$ as follow: $\mathcal{D} = \left\{ n \,\middle|\, Eft_1(t) = \frac{m}{n},\ t \in T, m, n \in \mathbb{N}_{>0} \right\} \cup \left\{ n \,\middle|\, Lft_1(t) = \frac{m}{n},\ t \in T, m, n \in \mathbb{N}_{>0} \right\} = \{2, 4\}$.

Let $c = LCM(\mathcal{D}) = LCM(2, 4) = 4$, $D_2(t_1) = \left[Eft_1(t_1) * c = 0 * 4, Lft_1(t_1) * c = \frac{1}{4} * 4 \right] = [0, 1]$, $D_2(t_2) = \left[Eft_1(t_2) * c = \frac{1}{4} * 4, Lft_1(t_2) * c = \frac{3}{4} * 4 \right] = [1, 3]$, and $D_2(t_3) = \left[Eft_1(t_3) * c = 0 * 4, Lft_1(t_3) * c = \frac{1}{2} * 4 \right] = [0, 2]$. Then, the time intervals of $\widetilde{\mathcal{TN}_2}$ have natural-valued boundaries. Moreover, $\widetilde{\mathcal{TN}_1}$ and $\widetilde{\mathcal{TN}_2}$ are time equivalent, because the static functions in the TPNs are proportional to the non-zero constant 4. As a consequence, for example, for the run $\sigma_1 = t_1 t_3 (0.25) t_2 (0.125) t_3$ of $\widetilde{\mathcal{TN}_1}$, there exists the run $\sigma_2 = t_1 t_3 (1) t_2 (0.5) t_3$ of $\widetilde{\mathcal{TN}_2}$, with time elapsings that are four longer times. Conversely, for the run σ_2 of $\widetilde{\mathcal{TN}_2}$, there is the run σ_1 of $\widetilde{\mathcal{TN}_1}$ with time elapsings that are four times shorter. ◻

So, if two time Petri nets are time equivalent, then each run of the one TPN can be obtained from some run of the other TPN with the same untimed part and with time elapsings multiplied by the proportionality constant. In other words, time in the one TPN goes slower (or faster) than in the other TPN. With that, it is worth stressing that the TPNs have the same behavioral properties (e.g., safeness, liveness, marking reachability, etc.).

In the sequel, we will consider only TPNs with time intervals having natural-valued boundaries.

3. State Space Discretization for $\mathcal{TN}$

In this section, we demonstrate that in the TPN for any run there exists a run with the same untimed part and with natural-value time (and even unit time) elapsings.

For the TPN $\mathcal{TN}$, let $\widehat{\mathcal{FS}}(\mathcal{TN})$ be the set of all runs from $\mathcal{FS}(\mathcal{TN})$ of the form: $\hat{\sigma} = \tau_0\, t_1\, \tau_1 \dots \tau_{k-1}\, t_k\, \tau_k$, i.e. time elapsings and transition firings alternate in the runs. So, in $\mathcal{TN}$ for the run $\hat{\sigma} \in \widehat{\mathcal{FS}}(\mathcal{TN})$, we have the following changes in states: $(M_0, I_0) \xrightarrow{\tau_0} (M_0, I_0') \xrightarrow{t_1} (M_1, I_1) \xrightarrow{\tau_1} (M_1, I_1') \dots (M_{k-1}, I_{k-1}) \xrightarrow{\tau_{k-1}} (M_{k-1}, I'_{k-1}) \xrightarrow{t_k} (M_k, I_k) \xrightarrow{\tau_k} (M_k, I_k')$. As we will see later (in the proof of Corollary 2), any run from $\mathcal{FS}(\mathcal{TN})$ can be represented in the above form. Define the set $\widehat{\mathcal{UFS}}(\mathcal{TN}) = \{Untimed(\hat{\sigma}) | \hat{\sigma} \in \widehat{\mathcal{FS}}(\mathcal{TN})\}$.

Next, for an untimed sequence from $\widehat{\mathcal{UFS}}(\mathcal{TN})$, we construct the parametric run (which is, in fact, a modification of a run, with variables instead of the time elapsing values) and a set of conditions on the values of these variables, by induction of the number of the variables. At each induction step, we define a prefix of the parametric run and the conditions for the values of its variables, increasing the run's length by one variable.

Definition 4. Let $\mathcal{TN} = ((P, T, F, M_0, L), D)$ be a time Petri net, $t_1 \dots t_k \in \widehat{\mathcal{UFS}}(\mathcal{TN})$ and $X = \{x_0, \dots, x_k\}$ be a set of variables. We construct a finite sequence of the tuples of the form $(\omega_i, B_{\omega_i}, M_{\omega_i}, I'_{\omega_i})$, by induction on $0 \leq i \leq k$.

$i = 0$. Then,
- $\omega_0 = x_0$;
- $B_{\omega_0} = \emptyset$;
- $M_{\omega_0} = M_0$;
- $I'_{\omega_0}(t) = x_0$, for all $t \in En(M_{\omega_0})$.

$i > 0$. Assume that $(\omega_{i-1}, B_{\omega_{i-1}}, M_{\omega_{i-1}}, I'_{\omega_{i-1}})$ is already constructed. Then,
- $\omega_i = \omega_{i-1} t_i x_i$;
- $B_{\omega_i} = B_{\omega_{i-1}} \cup \{Eft(t_i) \leq I'_{\omega_{i-1}}(t_i) \leq Lft(t_i)\}$;
- $M_{\omega_i} = (M_{\omega_{i-1}} \setminus \bullet t_i) \cup t_i \bullet$;

– $I'_{\omega_i}(t) = \begin{cases} x_i, & \text{if } \uparrow enabled(t, M_{\omega_{i-1}}, t_i), \\ I'_{\omega_{i-1}}(t) + x_i, & otherwise, \end{cases}$ for all $t \in En(M_{\omega_i})$.

Then, $\omega = \omega_k = x_0 t_1 x_1 \ldots t_k x_k$ is a *parametric run of* $\mathcal{TN}$, and $B_\omega = B_{\omega_k} \cup \{0 \le x_k < \infty\}$ is the *set of conditions on the values of the variables from X*.

We use $\mathcal{B}_\omega = \{I'_{\omega_{i-1}}(t_i) | (Eft(t_i) \le I'_{\omega_{i-1}}(t_i) \le Lft(t_i)) \in B_\omega, 1 \le i \le k\} \cup \{x_k\}$ to denote the *set of the variable parts of the inequalities from* B_ω.

Example 3. Contemplate the TPN $\widetilde{\mathcal{TN}} = ((P, T, F, M_0, L), D)$, shown in Fig. 1. For the transition sequence $Untimed(\hat{\sigma}) = t_1\, t_3\, t_2$, obtained from the run $\hat{\sigma} = (0.5)t_1(0.5)t_3(2.3)t_2(1.7)$ of $\widetilde{\mathcal{TN}}$, we construct the sequence of the following tuples $(\omega_i, B_{\omega_i}, M_{\omega_i}, I'_{\omega_i})$, with $0 \le i \le 3$.

– $i = 0$. Set $\omega_0 = x_0$; $B_{\omega_0} = \emptyset$; $M_{\omega_0} = M_0 = \{p_1, p_2\}$; $En(M_{\omega_0}) = \{t_1, t_3\}$; and $I'_{\omega_0}(t_1) = I'_{\omega_0}(t_3) = x_0$.
– $i = 1$. Set $\omega_1 = \omega_0 t_1 x_1 = x_0 t_1 x_1$;
  $B_{\omega_1} = B_{\omega_0} \cup \{Eft(t_1) = 0 \le I'_{\omega_0}(t_1) \le Lft(t_1) = 1\} = B_{\omega_0} \cup \{0 \le x_0 \le 1\}$;
  $M_{\omega_1} = (M_{\omega_0} \setminus \bullet t_1) \cup t_1 \bullet = \{p_3, p_2\}$ and $En(M_{\omega_1}) = \{t_3\}$;
  $I'_{\omega_1}(t_3) = I'_{\omega_0}(t_3) + x_1 = x_0 + x_1$, as $\uparrow enabled(t_3, M_{\omega_1}, t_1)$ is false.
– $i = 2$. Set $\omega_2 = \omega_1 t_3 x_2 = x_0 t_1 x_1 t_3 x_2$;
  $B_{\omega_2} = B_{\omega_1} \cup \{Eft(t_3) = 0 \le I'_{\omega_1}(t_3) \le Lft(t_3) = 2\} = B_{\omega_1} \cup \{0 \le x_0 + x_1 \le 2\}$;
  $M_{\omega_2} = (M_{\omega_1} \setminus \bullet t_3) \cup t_3 \bullet = \{p_3, p_4\}$ and $En(M_{\omega_2}) = \{t_2\}$;
  $I'_{\omega_2}(t_2) = x_2$, as $\uparrow enabled(t_2, M_{\omega_2}, t_3)$ is true.
– $i = 3$. *Set* $\omega_3 = \omega_2 t_2 x_3 = x_0 t_1 x_1 t_3 x_2 t_2 x_3$;
  $B_{\omega_3} = B_{\omega_2} \cup \{Eft(t_2) = 1 \le I'_{\omega_2}(t_2) \le Lft(t_2) = 3\} = B_{\omega_2} \cup \{1 \le x_2 \le 3\}$;
  $M_{\omega_3} = (M_{\omega_2} \setminus \bullet t_2) \cup t_2 \bullet = \{p_1, p_2\}$ and $En(M_{\omega_3}) = \{t_1, t_3\}$;
  $I'_{\omega_3}(t_1) = I'_{\omega_3}(t_3) = x_3$, as $\uparrow enabled(t_1, M_{\omega_3}, t_2)$ and $\uparrow enabled(t_3, M_{\omega_3}, t_2)$ are true.

Then, the parametric run has the form $\omega = \omega_3 = x_0 t_1 x_1 t_3 x_2 t_2 x_3$, where $X = \{x_0, \ldots, x_k\}$ is the set of the real variables. Moreover, it holds that

$$B_\omega = B_{\omega_3} = \begin{Bmatrix} 0 \le x_0 \le 1, \\ 0 \le x_0 + x_1 \le 2, \\ 1 \le x_2 \le 3, \\ 0 \le x_3 < \infty \end{Bmatrix}, \text{ and}$$

$$\mathcal{B}_\omega = \{x_0, x_0 + x_1, x_2, x_3\}.$$

□

A function $\beta: X = \{x_0, \ldots, x_k\} \to \mathbb{R}_{\ge 0}$ is called *assignment of* ω. We write $[\omega]_\beta$ ($[I'_{\omega_{i-1}}(t_i)]_\beta$) for a parametric run ω (for the value of a linear function $I'_{\omega_{i-1}}(t_i) \in \mathcal{B}_\omega$) under the assignment β.The mapping β is a *solution* of B_ω iff $Eft(t_i) \le [I'_{\omega_{i-1}}(t_i)]_\beta \le Lft(t_i)$, for all $I'_{\omega_{i-1}}(t_i) \in \mathcal{B}_\omega$.

Example 4. Consider the parametric run $\omega = x_0 t_1 x_1 t_3 x_2 t_2 x_3$, the set $\mathcal{B}_\omega = \{x_0, x_0 + x_1, x_2, x_3\}$, from Example 3, and an assignment $\beta: \{x_0, \ldots, x_3\} \to \mathbb{R}_{\ge 0}$ such that $\beta(x_0) = 0.7, \beta(x_1) = 0.3, \beta(x_2) = 1.4$, $\beta(x_3) = 2$. Then, we obtain that $[\omega]_\beta = (0.7)t_1(0.3)t_3(1.4)t_2(2)$. Moreover, we get: $Eft(t_1) = 0 \le [x_0]_\beta = 0.7 \le Lft(t_1) = 1$, $Eft(t_3) = 0 \le [x_0 + x_1]_\beta = 1 \le Lft(t_3) = 2$, $Eft(t_2) = 1 \le [x_2]_\beta = 1.4 \le Lft(t_2) = 3$. Therefore, β is a solution of B_ω.
□

We next establish that any solution of B_ω maps a parametric run ω of the TPN to its run.

Lemma 1. Let $\mathcal{TN}$ be a TPN, $\omega = x_0 t_1 x_1 \ldots t_k x_k$ be a parametric run of $\mathcal{TN}$, and B_ω be the set of conditions on the values of the variables $x_0, x_1, \ldots, x_k$. If β is a solution of B_ω, then $[\omega]_\beta \in \widehat{\mathcal{FS}}(\mathcal{TN})$. Moreover, for any $I'_{\omega_{i-1}}(t_i) \in \mathcal{B}_\omega$, $[I'_{\omega_{i-1}}(t_i)]_\beta$ is the value of $I'_{i-1}(t_i)$ in the run $[\omega]_\beta$, when $\mathcal{TN}$ functions along the run.

Proof. See Appendix. □

Next, for an arbitrary run $\hat{\sigma}$ from $\widehat{\mathcal{FS}}(\mathcal{TN})$ with real-value time elapsings, we construct a natural-value assignment β_ω to the variables in the corresponding parametric run ω of $\mathcal{TN}$, by induction on the number of the variables in ω (i.e. the number of time elapsings in the run $\hat{\sigma}$). Starting from the end of the run $\hat{\sigma}$, at each induction step, we round, the value of the corresponding time elapsing of $\hat{\sigma}$, down or up to a natural number nearest to the value and agreed with the values of the other time elapsings in $\hat{\sigma}$.

Definition 5. Let $\mathcal{TN}$ be a TPN, $\tau_0 t_1 \tau_1 \ldots t_k \tau_k \in \widehat{\mathcal{FS}}(\mathcal{TN})$, $\omega = x_0 t_1 x_1 \ldots t_k x_k$ be the parametric run of $\mathcal{TN}$, B_ω be the set of conditions on the values of the variables from $X = \{x_0, \ldots, x_k\}$, and $\mathcal{B}_\omega$ be the set of the variable parts of the inequalities from B_ω. We construct a sequence of functions $\beta_i: X \to \mathbb{R}_{\ge 0}$ by induction on $0 \le i \le k$.

$i = 0$. Then, for all $0 \le j \le k$,

$$\beta_0(x_j) = \begin{cases} \lfloor \tau_j \rfloor & \text{if } j = k, \\ \tau_j & \text{otherwise} \end{cases}.$$

$i > 0$. In the construction of β_i, we use auxiliary functions defined for all $0 \le j \le k$ as follows:

$$\underline{\beta_i}(x_j) = \begin{cases} \lfloor \beta_{i-1}(x_j) \rfloor & \text{if } j = k - i, \\ \beta_{i-1}(x_j) & \text{otherwise;} \end{cases} \text{ and } \overline{\beta_i}(x_j) = \begin{cases} \lceil \beta_{i-1}(x_j) \rceil & \text{if } j = k - i, \\ \beta_{i-1}(x_j) & \text{otherwise.} \end{cases}$$

If $\exists\, I'_{\omega_{l-1}}(t_l) \in \mathcal{B}_\omega$ $(1 \le l \le k)$ such that $\lceil [I'_{\omega_{l-1}}(t_l)]_{\underline{\beta_i}} \rceil < \lfloor [I'_{\omega_{l-1}}(t_l)]_{\beta_0} \rfloor$, then $\beta_i = \overline{\beta_i}$, else $\beta_i = \underline{\beta_i}$.

Define a natural-value assignment $\beta_\omega: X \to \mathbb{N}$ as follows: $\beta_\omega = \beta_k$.

Example 5. Consider the TPN $\widetilde{\mathcal{TN}}$ from Example 1, the run $\hat{\sigma} = (0.5)t_1(0.5)t_3(2.3)t_2(1.7) \in \widehat{\mathcal{FS}}(\widetilde{\mathcal{TN}})$, the parametric run $\omega = x_0 t_1 x_1 t_3 x_2 t_2 x_3$ and the set $\mathcal{B}_\omega = \{x_0, x_0 + x_1, x_2, x_3\}$ from Example 3. Using Definition 5, construct the sequence of the assignments β_i by induction on $0 \le i \le 3$.

– $i = 0$. We set $\beta_0(x_j)$, for all $0 \le j \le k = 3$, as follows: $\beta_0(x_0) = \tau_0 = 0.5$, $\beta_0(x_1) = \tau_1 = 0.5$, $\beta_0(x_2) = \tau_2 = 2.3$, and $\beta_0(x_3) = \lfloor \tau_3 = 1.7 \rfloor = 1$, because $j = k = 3$.

– $i = 1$. Construct auxiliary functions $\underline{\beta_1}(x_j)$ and $\overline{\beta_1}(x_j)$, for all $0 \le j \le k = 3$, as follows: $\underline{\beta_1}(x_0) = \overline{\beta_1}(x_0) = \beta_0(x_0) = 0.5$, $\underline{\beta_1}(x_1) = \overline{\beta_1}(x_1) = \beta_0(x_1) = 0.5$, $\underline{\beta_1}(x_3) = \overline{\beta_1}(x_3) = \beta_0(x_3) = 1$, and $\underline{\beta_1}(x_2) = \lfloor \beta_0(x_2) \rfloor = 2$, $\overline{\beta_1}(x_2) = \lceil \beta_0(x_2) \rceil = 3$, because $j = k - i = 2$.

  Moreover, we get: $\lceil [x_0]_{\underline{\beta_1}} \rceil = 1 \ge \lfloor [x_0]_{\beta_0} \rfloor = 0$, $\lceil [x_0 + x_1]_{\underline{\beta_1}} \rceil = 1 \ge \lfloor [x_0 + x_1]_{\beta_0} \rfloor = 1$, $\lceil [x_2]_{\underline{\beta_1}} \rceil = 2 \ge \lfloor [x_2]_{\beta_0} \rfloor = 2$. Then, $\beta_1 = \underline{\beta_1}$.

– $i = 2$. Construct $\underline{\beta_2}(x_j)$ and $\overline{\beta_2}(x_j)$, with $0 \le j \le 3$, as follows: $\underline{\beta_2}(x_0) = \overline{\beta_2}(x_0) = \beta_1(x_0) =$

0.5, $\underline{\beta_2}(x_2) = \overline{\beta_2}(x_2) = \beta_1(x_2) = 2$, $\underline{\beta_2}(x_3) = \overline{\beta_2}(x_3) = \beta_1(x_3) = 1$ and $\underline{\beta_2}(x_1) = \lfloor \beta_1(x_1) \rfloor = 0$, $\overline{\beta_2}(x_1) = \lceil \beta_1(x_1) \rceil = 1$, because $j = k - i = 1$.

Moreover, we get: $\lceil [x_0]_{\underline{\beta_2}} \rceil = 1 \geq \lfloor [x_0]_{\beta_0} \rfloor = 0$, $\lceil [x_0 + x_1]_{\underline{\beta_2}} \rceil = 1 \geq \lfloor [x_0 + x_1]_{\beta_0} \rfloor = 1$, $\lceil [x_2]_{\underline{\beta_2}} \rceil = 2 \geq \lfloor [x_2]_{\beta_0} \rfloor = 2$. Then, $\beta_2 = \underline{\beta_2}$.

– $i = 3$. Construct $\underline{\beta_3}(x_j)$ and $\overline{\beta_3}(x_j)$, with $0 \leq j \leq 3$, as follows: $\underline{\beta_3}(x_1) = \overline{\beta_3}(x_1) = \beta_2(x_1) = 0$, $\underline{\beta_3}(x_2) = \overline{\beta_3}(x_2) = \beta_2(x_2) = 2$, $\underline{\beta_3}(x_3) = \overline{\beta_3}(x_3) = \beta_2(x_3) = 1$ and $\underline{\beta_3}(x_0) = \lfloor \beta_2(x_0) \rfloor = 0$, $\overline{\beta_3}(x_0) = \lceil \beta_2(x_0) \rceil = 1$, because $j = k - i = 0$.

Moreover, we have $\lceil [x_0]_{\underline{\beta_3}} \rceil = 0 \geq \lfloor [x_0]_{\beta_0} \rfloor = 0$, $\lceil [x_0 + x_1]_{\underline{\beta_3}} \rceil = 0 < \lfloor [x_0 + x_1]_{\beta_0} \rfloor = 1$, $\lceil [x_2]_{\underline{\beta_3}} \rceil = 2 \geq \lfloor [x_2]_{\beta_0} \rfloor = 2$. Then, there exists $I'_{\omega_1}(t_3) \in \mathcal{B}_\omega$ such that $\lceil [I'_{\omega_1}(t_3)]_{\underline{\beta_3}} \rceil < \lfloor [I'_{\omega_1}(t_3)]_{\beta_0} \rfloor$. Therefore, $\beta_3 = \overline{\beta_3}$.

Then, $\beta_\omega = \beta_3$, and we obtain the sequence $[\omega]_{\beta_\omega} = (1)t_1(0)t_3(2)t_2(1)$. □

Next, we show that the assignment β_ω is a solution of ω, i.e. β_ω satisfies the inequalities from B_ω.

Proposition 1. β_ω is a solution of B_ω.

Proof. See Appendix. □

Thanks to Lemma 1 and Proposition 1, the theorem below follows immediately.

Theorem 2. Let $\mathcal{TN}$ be a time Petri net and $\omega = x_0 t_1 x_1 \ldots t_k x_k$ be a parametric run of $\mathcal{TN}$. Then, there exists a mapping $\beta_\omega : X = \{x_0, \ldots, x_k\} \to \mathbb{N}$ such that $[\omega]_{\beta_\omega} \in \widehat{\mathcal{FS}}(\mathcal{TN})$.

Proof. Consider the mapping $\beta_\omega : X = \{x_0, \ldots, x_k\} \to \mathbb{N}$ from Definition 5. By Proposition 1, β_ω is a solution of B_ω, and, moreover, $[\omega]_{\beta_\omega} \in \widehat{\mathcal{FS}}(\mathcal{TN})$, due to Lemma 1. □

We are now ready to show that in the TPN for any run, there exists a run with the same untimed part and with unit time elapsings.

Corollary 2. Let $\mathcal{TN}$ be a time Petri net and $\sigma \in \mathcal{FS}(\mathcal{TN})$. Then, there is $\sigma' \in \mathcal{FS}(\mathcal{TN})$ with unit-value time elapsings such that $Untimed(\sigma) = Untimed(\sigma')$.

Proof. Due to Definition 2, we obtain the following properties for time elapsings:

a) $S \xrightarrow{0} S$;

b) if $S \xrightarrow{\tau} S'$ and $S' \xrightarrow{\tau'} S''$, with $\tau, \tau' \in \mathbb{R}_{\geq 0}$, then $S \xrightarrow{\tau+\tau'} S''$;

c) if $S \xrightarrow{\tau} S'$, then for every $\tau', \tau'' \in \mathbb{R}_{\geq 0}$ such that $\tau = \tau' + \tau''$, $S \xrightarrow{\tau'} S'' \xrightarrow{\tau''} S'$ for some S''.

By items a) and b), there exists $\hat{\sigma} \in \widehat{\mathcal{FS}}(\mathcal{TN})$ such that $Untimed(\sigma) = Untimed(\hat{\sigma})$, thanks to σ being a finite sequence. Due to Theorem 2, there is a run $\widehat{\sigma'} \in \widehat{\mathcal{FS}}(\mathcal{TN})$ with natural-value time elapsings such that $Untimed(\hat{\sigma}) = Untimed(\widehat{\sigma'})$. Thanks to c), we can construct $\sigma' \in \mathcal{FS}(\mathcal{TN})$ with unit time elapsings such that $Untimed(\widehat{\sigma'}) = Untimed(\sigma')$. Therefore, we obtain that $Untimed(\sigma) = Untimed(\sigma')$.
□

Thanks to Corollary 2, in the sequel, we will consider time Petri nets with unit time elapsings (denoted by $\surd$).

4. Time Processes of $\mathcal{TN}$

In this section, the concept of causality-based net process is presented and studied in the context of TPNs with weak semantics. We start with definitions related to casual nets that contain events and conditions connected by causal dependence and concurrency (absence of causality).

Definition 6. A (*labeled over* $Act \cup \{tick\}$) *casual net* is a finite, acyclic net $TN = (B, E, G, l)$ with a set B of conditions*;* a set E of events*;* a flow relation $G \subseteq (B \times E) \cup (E \times B)$ such that $|b\bullet| \leq 1 \land |\bullet b| \leq 1$, for all $b \in B$; a labeling function $l : E \to Act \cup \{tick\}$.

Informally speaking, the «tick» label means a clock ticking.
Casual nets $TN = (B, E, G, l)$ and $TN' = (B', E', G', l')$ are called *isomorphic* (denoted $TN \simeq TN'$) iff there exists a bijective mapping $\gamma : B \cup E \to B' \cup E'$ such that:

– $\gamma(B) = B'$ and $\gamma(E) = E'$;
– $x\, G\, y \Leftrightarrow \gamma(x)\, G'\, \gamma(y)$, for all $x, y \in B \cup E$;
– $l(e) = l'(\gamma(e))$, for all $e \in E$.

Introduce auxiliary notions and notations for the casual net $TN = (B, E, G, l)$.
The set $\bullet b$ ($b\bullet$) is associated with a single event, for any $b \in B$. Let $\bullet TN = \{b \in B \mid \bullet b = \emptyset\}$.
Define $\prec = G^+$, $\preccurlyeq = G^*$ (*causality*). A subset $E' \subseteq E$ is a *downward closed set of events* iff $e \in E'$ implies $e' \in E'$, for all $e' \prec e$. In this case, $Cut(E') = (\bullet TN \cup E'\bullet) \setminus \bullet E'$.
A subset $B' \subseteq B$ is a *co-set* (*a subset of concurrent conditions*) iff $\neg(b \prec b')$ and $\neg(b' \prec b)$, for all $b, b' \in B'$. A *cut* is a maximal (w.r.t. set inclusion) co-set.
A sequence $\rho = e_1 \ldots e_n$ $(n \geq 0)$ of events is a *linearization* of TN [5] if each event of TN appears in the sequence exactly once, and the following holds: $e_i \prec e_j \Rightarrow i < j$, for all $1 \leq i, j \leq n$. For a linearization $\rho = e_1 \ldots e_n$ $(n \geq 0)$ of TN, define the following:
– ρ_0 is the empty sequence and $\rho_k = e_1 \ldots e_k$ $(1 \leq k \leq n)$;
– $E_0 = \emptyset$ and $E_k = \bigcup_{1 \leq i \leq k} e_i$ $(1 \leq k \leq n)$.
By the construction of the linearization, E_k is a downward closed set of events, for all $0 \leq k \leq n$. As we will see later (in Lemma 2), $Cut(E_k)$ is a cut, for all $1 \leq k \leq n$.
Informally speaking, a linearization is an interleaving representation of a "computation" of TN and the value of the function Cut of any prefix of the linearization is a "marking" of TN, reachable after occurring the events from the prefix.

Example 6. Fig. 2 shows a causal net $\widetilde{TN} = (B, E, G, l)$, with $B = \{b_1, \ldots, b_{10}\}$; $E = \{e_1, \ldots, e_5\}$; $G = \{(b_1, e_1), (e_1, b_3), (b_3, e_2), (b_2, e_2), \ldots, (e_5, b_9), (e_5, b_{10})\}$; $l(e_1) = l(e_3) = b$, $l(e_2) = l(e_4) = tick$, $l(e_5) = a$. We see that $|\bullet b| \leq 1 \land |b\bullet| \leq 1$, for all $b \in B$, and $\bullet\widetilde{TN} = \{b_1, b_2\}$. Clearly, $e_1 \prec e_2 \prec e_3 \prec e_4 \prec e_5$. Moreover, $\{b_1, b_2\}, \{b_2, b_3\}, \ldots, \{b_9, b_{10}\}$ are cuts in $\widetilde{TN}$. It is easy to check that $\tilde{\rho} = e_1, e_2, e_3, e_4, e_5$ is a linearization of $\widetilde{TN}$, because each event of $\widetilde{TN}$ appears in the sequence exactly once, and if $e_i \prec e_j$, then $i < j$, for all $1 \leq i, j \leq 5$. Define the downward closed sets $E_i = \bigcup_{1 \leq j \leq k} e_j$ and the sets $Cut(E_i) = (\bullet TN \cup E_i\bullet) \setminus \bullet E_i = (\{b_1, b_2\} \cup E_i\bullet) \setminus \bullet E_i$, for all $0 \leq i \leq 5$, as follows:

– $E_0 = \emptyset$, $Cut(E_0) = (\{b_1, b_2\} \cup \emptyset) \setminus \emptyset = \{b_1, b_2\}$;
– $E_1 = \{e_1\}$, $Cut(E_1) = (\{b_1, b_2\} \cup \{b_3\}) \setminus \{b_1\} = \{b_2, b_3\}$;
– $E_2 = \{e_1, e_2\}$, $Cut(E_2) = (\{b_1, b_2\} \cup \{b_3, b_4, b_5\}) \setminus \{b_1, b_2, b_3\} = \{b_4, b_5\}$;
– $E_3 = \{e_1, e_2, e_3\}$, $Cut(E_3) = (\{b_1, b_2\} \cup \{b_3, b_4, b_5, b_6\}) \setminus \{b_1, b_2, b_3, b_5\} = \{b_4, b_6\}$;
– $E_4 = \{e_1, e_2, e_3, e_4\}$, $Cut(E_4) = (\{b_1, b_2\} \cup \{b_3, b_4, b_5, b_6, b_7, b_8\}) \setminus \{b_1, b_2, b_3, b_4, b_5, b_6\} =$

$\{b_7, b_8\}$;

– $E_5 = \{e_1, e_2, e_3, e_4, e_5\}$, $Cut(E_5) = B \setminus \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\} = \{b_9, b_{10}\}$. □

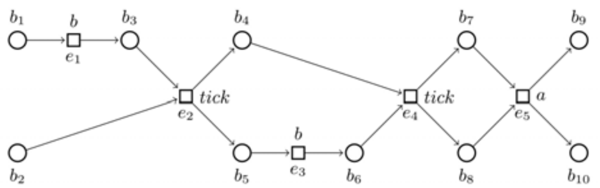

Fig. 2. *A casual net* $\widetilde{TN}$

Proposition 2. Any casual net TN has a linearization $\rho = e_1 \dots e_n$.

Proof. Take the maximal (w.r.t. set inclusion) set $min(E) \subseteq E$ such that if $e \in min(E)$, then $\neg(e' \prec e)$, for all $e' \in E$. After removing an event e_{min} from E, we get that $min(E') = min(E) \setminus e_{min}$ or $min(E') = (min(E) \setminus e_{min}) \cup (e_{min}\bullet)\bullet$, where $E' = E \setminus e_{min}$. Construct a sequence $\rho = e_1 \dots e_n$ of events, by selecting a minimal event and removing it from E, at each step. By the construction, any event of TN appears in the sequence exactly once and $e_i \prec e_j \Rightarrow i < j$, for all $1 \le i, j \le n$, due to TN being a causal net. □

The results of the below lemma will be useful to establish the relationships between "markings" (values of the function Cut) of a causal net TN and markings of a time Petri net $\mathcal{TN}$, and between linearizations of TN and runs of $\mathcal{TN}$, when we construct partial order semantic for time Petri nets.

Lemma 2. Let TN be a casual net and $\rho = e_1 \dots e_n$ its linearization. Then, it holds:

a) $Cut(E_k) = (Cut(E_{k-1}) \setminus \bullet e_k) \cup e_k\bullet$, for all $1 \le k \le n$;

b) $\bullet e_k \subseteq Cut(E_{k-1})$, for all $1 \le k \le n$;

c) if $Cut(E_k) \ne \emptyset$, then $Cut(E_k)$ is a cut of TN, for all $0 \le k \le n$.

Proof. See Appendix. □

Next, we introduce a notion of a homomorphism from a causal net to a time Petri net, in order to define the concept of time processes of the time Petri net.

Definition 7. Let $\mathcal{TN} = ((P, T, F, M_0, L), D)$ be a TPN and $TN = (B, E, G, l)$ be a casual net. A *homomorphism*[1] from TN to $\mathcal{TN}$ is a mapping $\varphi : (B \cup E) \to (P \cup T \cup \{\surd\})$ such that it holds the following:

– $\varphi(B) \subseteq P$ and $\varphi(E) \subseteq (T \cup \{\surd\})$;
– for all $e \in E$ such that $\varphi(e) \in T$,
  - the restriction of φ to $\bullet e$ is a bijection between $\bullet e$ and $\bullet\varphi(e)$,
  - the restriction of φ to $e\bullet$ is a bijection between $e\bullet$ and $\varphi(e)\bullet$;
– for all $e \in E$ such that $\varphi(e) = \surd$,

[1] In fact, φ is a homomorphism from TN to $\mathcal{TN}' = ((P, T, F', M_0, L'), D)$, where $F' \subseteq (P \times T \cup \{\surd\}) \cup (T \cup \{\surd\} \times P)$ such that $F' = F \cup \{(p, \surd), (\surd, p) \mid p \in P\}$, and $L' : T \cup \{\surd\} \to Act \cup \{tick\}$, such that $L'(x) = \begin{cases} L(x), \text{if } x \in T \\ tick, \text{otherwise} \end{cases}$. In this case, we can see that φ is a structure-preserving mapping. However, following the traditions of terms and definitions in the literature on TPNs, we omit this construction of $\mathcal{TN}'$.

  - the restriction of φ to $\bullet e$ and the restriction of φ to $e\bullet$ are injections,
  - $\bullet e$ and $e\bullet$ are cuts of TN and $\varphi(\bullet e) = \varphi(e\bullet)$;
– the restriction of φ to $\bullet TN$ is a bijection between $\bullet TN$ and M_0;
– $l(e) = \begin{cases} L(\varphi(e)), & \text{if } \varphi(e) \in T, \\ tick, & \text{otherwise} \end{cases}$, for all $e \in E$.

Then, the pair $\pi = (TN, \varphi)$ is called *a time process of* $\mathcal{TN}$.

Time processes $\pi = (TN, \varphi) = ((B, E, G, l), \varphi)$ and $\pi' = (TN', \varphi') = ((B', E', G', l'), \varphi')$ of $\mathcal{TN}$ are *isomorphic* (denoted $\pi \simeq \pi'$) if there is an isomorphism $\gamma : TN \simeq TN'$ such that $\varphi(x) = \varphi'(\gamma(x))$, for all $x \in B \cup E$.

Example 7. Consider the TPN $\widetilde{\mathcal{TN}}$ depicted in Fig. 1, the casual net $\widetilde{TN}$ shown in Fig. 2, and a mapping φ defined as follows: $\varphi(b_1) = \varphi(b_9) = p_1$, $\varphi(b_2) = \varphi(b_5) = \varphi(b_{10}) = p_2$, $\varphi(b_3) = \varphi(b_4) = \varphi(b_7) = p_3, \varphi(b_6) = \varphi(b_8) = p_4, \varphi(e_1) = t_1$, $\varphi(e_5) = t_2, \varphi(e_3) = t_3$, $\varphi(e_2) = \varphi(e_4) = \surd$; Then, we have that $\varphi(B) \subseteq P$, $\varphi(E) \subseteq (T \cup \{\surd\})$, and the restriction of φ *to* $\bullet TN = \{b_1, b_2\}$ is a bijection between $\{b_1, b_2\}$ and $M_0 = \{p_1, p_2\}$. Moreover, it holds the following: for all $e \in E$ such that $\varphi(e) \in T$, the restriction of φ to •e ($e\bullet$) is a bijection between •e ($e\bullet$) and •φ(e) ($\varphi(e)\bullet$); and for all $e \in E$ such that $\varphi(e) = \surd$, the restriction of φ to $\bullet e$ ($e\bullet$) is an injection, $\bullet e$ and $e\bullet$ are cuts of TN and $\varphi(\bullet e) = \varphi(e\bullet)$. For example, consider the events e_1 and e_2. We know that $\varphi(e_1) = t_1$ and $\varphi(e_2) = \surd$. The restriction of φ to $\bullet e_1 = \{b_1\}$ ($e_1\bullet = \{b_3\}$) is a bijection between $\{b_1\}$ ($\{b_3\}$) and $\varphi(b_1) = \{p_1\}$ ($\varphi(b_3) = \{p_3\}$). Furthermore, the restriction of φ to $\bullet e_2 = \{b_2, b_3\}$ ($e_2\bullet = \{b_4, b_5\}$) is an injection, $\{b_2, b_3\}$ and $\{b_4, b_5\}$ are cuts of TN and $\varphi(\{b_2, b_3\}) = \varphi(\{b_5, b_4\}) = \{p_2, p_3\}$. We see that $l(e) = \begin{cases} L(\varphi(e)), \text{if } \varphi(e) \in T \\ \text{tick, otherwise} \end{cases}$, for all $e \in E$. Therefore, φ is the homomorphism from $\widetilde{TN}$ to $\widetilde{\mathcal{TN}}$, and, hence, $\pi = (\widetilde{TN}, \varphi)$ is a time process of $\widetilde{\mathcal{TN}}$. □

For a time process $\pi = (TN, \varphi)$ of $\mathcal{TN}$, we introduce the function Age that defines "age" of each condition b of TN. More specifically, if $b \in \bullet TN$ is an input condition in TN, i.e. the place $\varphi(b)$ contains a token at the initial marking of $\mathcal{TN}$, then the "age" of b is equal to 0. Also, if b is an output condition of an event e that corresponds to the firing of the transition $\varphi(e)$ of $\mathcal{TN}$, i.e. the place $\varphi(b)$ of $\mathcal{TN}$ got a token immediately after the firing of $\varphi(e)$, then the "age" of b is equal to 0. Otherwise, i.e. if b is an output condition of an event e that corresponds to time elapsing, then the "age" of b is the increased by 1 "age" of the input condition b' of e, such that $\varphi(b) = \varphi(b')$, i.e. b and b' match in the same place of $\mathcal{TN}$.

$$Age(b) = \begin{cases} 0, & if\ b \in \bullet TN \vee (b \in e\bullet, \varphi(e) \in T), \\ Age(b') + 1, & if\ b \in e\bullet, \varphi(e) = \surd, b' \in \bullet e, \varphi(b') = \varphi(b). \end{cases}$$

Notice that if $b \in e\bullet$ and $\varphi(e) = \surd$, then there exists the only one condition $b' \in \bullet e$ such that $\varphi(b') = \varphi(b)$, due to Definition 7. In this case, the definition of the function Age is correct.
Informally speaking, the function Age matches each condition b of TN to the amount of time that has elapsed since the corresponding place $\varphi(b)$ of $\mathcal{TN}$ got a token, when $\mathcal{TN}$ progresses.

For a co-set B' of conditions of TN and a transition t of $\mathcal{TN}$ such that t is enabled at the marking $\varphi(B')$, determine the function **Clock** whose value is equal to the minimum "age" of the conditions from B', that correspond the input places of t.

$$\mathbf{Clock}(B', t) = \begin{cases} \bot, & if\ \bullet t = \emptyset, \\ \min\{Age(b) \mid \varphi(b) \in \bullet t, b \in B'\}, & otherwise. \end{cases}$$

Informally speaking, the function **Clock** matches B' and t to the amount of time that has elapsed since a token from the marking $\varphi(B')$ appeared in the last input place of the transition t, i.e. since the transition t became enabled at $\varphi(B')$. Later, we will establish a correspondence between the function **Clock** and the dynamic timing function I, when the TPN progresses.

We are now ready to introduce the concept of an admissible (correct) time process of $\mathcal{TN}$.

Definition 8. Let $\mathcal{TN}$ be a time Petri net. A time process $\pi = (TN, \varphi)$ of $\mathcal{TN}$ is *admissible* iff for all $e \in E$ it holds:

$$\varphi(e) \in T \Rightarrow Eft(\varphi(e)) \leq \mathbf{Clock}(\bullet e, \varphi(e)) \leq Lft(\varphi(e)).$$

Example 8. Verify that the time process π from Example 7 of $\widetilde{\mathcal{TN}}$ shown in Fig. 1 is admissible. By definitions, we have:

- $\mathbf{Clock}(\bullet e_1, \varphi(e_1)) = Age(b_1) = 0;$
- $\mathbf{Clock}(\bullet e_3, \varphi(e_3)) = Age(b_5) = Age(b_2) = 0 + 1 = 1;$
- $\mathbf{Clock}(\bullet e_5, \varphi(e_5)) = \min\{Age(b_7), Age(b_8)\} = \min\{Age(b_4) + 1, Age(b_6) = 0 + 1\} = \min\{(Age(b_3) = 0 + 1) + 1, 1\} = 1.$

Then, we obtain:

- $Eft(t_1) = 0 \leq \mathbf{Clock}(\bullet e_1, \varphi(e_1) = t_1) \leq 1 = Lft(\varphi(e_1)),$
- $Eft(t_3) = 0 \leq \mathbf{Clock}(\bullet e_3, \varphi(e_3) = t_3) \leq 2 = Lft(t_3),$
- $Eft(t_2) = 1 \leq \mathbf{Clock}(\bullet e_5, \varphi(e_5) = t_2) \leq 3 = Lft(t_2).$

So, π is an admissible time process of $\widetilde{\mathcal{TN}}$. □

5. Relating Runs and Time Processes of $\mathcal{TN}$

In this section, relationships between runs and linearizations (computations) of admissible time processes are investigated, in the context of time Petri nets. For this purpose, we define a mapping FS from a linearization $\rho = e_1 \dots e_n$ of a time process $\pi = (TN, \varphi)$ of the TPN $\mathcal{TN}$ to the sequence of the form: $FS(\rho) = \varphi(e_1) \dots \varphi(e_n)$. Here, TN is a causal net and φ is a homomorphism from TN to $\mathcal{TN}$.

First, we prove that if FS maps a prefix ρ_i $(0 \leq i \leq n)$ of the linearization ρ to the run of $\mathcal{TN}$ and (M_i, I_i) is the state reachable by the run, then φ maps the value of the function Cut of this prefix to the marking M_i. Moreover, for any transition t enabled at M_i, the value of the dynamic timing function $I_i(t)$ is equal to $\mathbf{Clock}(Cut(E_i), t)$, where E_i is the set of events from ρ_i.

Lemma 3. Let $\pi = (TN, \varphi)$ be a time process of the TPN $\mathcal{TN}$ and $\rho = e_1 \dots e_n$ be a linearization of TN. If $FS(\rho_i) = \varphi(e_1) \dots \varphi(e_i)$ is the run of $\mathcal{TN}$ from (M_0, I_0) to (M_i, I_i) for some $0 \leq i \leq n$, then it holds:

a) the restriction of φ to $Cut(E_i)$ is a bijection between $Cut(E_i)$ and M_i;
b) $\mathbf{Clock}(Cut(E_i), t) = I_i(t)$, for all $t \in En(M_i)$;
c) if $i < n$ and $\varphi(e_{i+1}) \in En(M_i)$, then $\mathbf{Clock}(\bullet e_{i+1}, \varphi(e_{i+1}) = I_i(\varphi(e_{i+1}))$.

Proof. See Appendix. □

We are now ready to establish an important property of the FS mapping – any linearization of a time process of the TPN is mapped to its runs.

Theorem 3. Given an admissible time process $\pi = (TN, \varphi)$ of $\mathcal{TN}$ and a linearization $\rho = e_1 \dots e_n$ of TN, $FS(\rho)$ is a run of $\mathcal{TN}$.

Proof. We shall prove by induction on $0 \leq i \leq n$ that $FS(\rho_i)$ is a run of $\mathcal{TN}$.

$i = 0$. Then, $FS(\rho_0)$ is the empty run.

$i > 0$. By the induction hypothesis, $FS(\rho_{i-1})$ is a run of $\mathcal{TN}$. If $\varphi(e_i) = \surd$, then $FS(\rho_i)$ is a run of $\mathcal{TN}$, due to Definition 2. Assume $\varphi(e_i) \in T$. By Lemma 2(b), we have $\varphi(\bullet e_i) \subseteq \varphi(Cut(E_{i-1}))$. As the restriction of φ to $\bullet e$ is a bijection between $\bullet e$ and $\bullet\varphi(e)$, we obtain $\bullet\varphi(e_i) \subseteq \varphi(Cut(E_{i-1}))$. Thus, we get $\varphi(e_i) \in En(M_{i-1})$, due to Lemma 3(a). Thanks to Lemma 3(c), we have that $I_{i-1}(\varphi(e_i)) = \mathbf{Clock}(\bullet e_i, \varphi(e_i))$. Then, it holds that $Eft(\varphi(e_i)) \leq I_{i-1}(\varphi(e_i)) \leq Lft(\varphi(e_i))$, by Definition 8. Therefore, $\varphi(e_i)$ can fire from (M_{i-1}, I_{i-1}), i.e. $FS(\rho_i)$ is a run of $\mathcal{TN}$. □

Next, we show that the mapping FS is a surjection, i.e. for an arbitrary run σ of $\mathcal{TN}$, there is exists an admissible time process $\pi^* = (TN^*, \varphi^*)$ of $\mathcal{TN}$ and a linearization ρ^* of TN^* such that $FS(\rho^*) = \sigma$. The following definition provides constructions of π^* and ρ^*.

Definition 9. Let $\mathcal{TN} = ((P, T, F, M_0, L), D)$ be a time Petri net and $\sigma = \overline{t_1} \dots \overline{t_n} \in (T \cup \{\surd\})^n$ be a run of $\mathcal{TN}$.

We construct a finite sequence of tuples (E_i, B_i, G_i, C_i) by induction on $0 \leq i \leq n$.

$i = 0$. Then, set:

- $E_0 = \emptyset;$
- $B_0 = \{b_{0,p} \mid p \in M_0\};$
- $G_0 = \emptyset;$
- $C_0 = B_0;$

$i > 0$. Assume that $(E_{i-1}, B_{i-1}, G_{i-1}, C_{i-1})$ is already constructed. Then, set:

- $E_i = E_{i-1} \cup \{e_i\};$
- $B_i = B_{i-1} \cup \begin{cases} \{b_{i,p} \mid b_{j,p} \in C_{i-1}\}, & if\ \overline{t_i} = \surd, \\ \{b_{i,p} \mid p \in \overline{t_i}\bullet\}, & otherwise \end{cases};$
- $G_i = G_{i-1} \cup \{(e_i, b_{i,p}) \mid b_{i,p} \in B_i\} \cup \begin{cases} \{(b_{j,p}, e_i) \mid b_{j,p} \in C_{i-1}\}, & if\ \overline{t_i} = \surd, \\ \{(b_{j,p}, e_i) \mid b_{j,p} \in C_{i-1},\ p \in \bullet\overline{t_i}\}, & otherwise \end{cases};$
- $C_i = (C_{i-1} \setminus \bullet e_i) \cup e_i\bullet.$

Define $\pi^* = (TN^*, \varphi^*) = ((B, E, G, l^*), \varphi^*)$ as follows:

- $B = B_n, E = E_n, G = G_n;$
- $\varphi^*(e_i) = \overline{t_i}$, for all $e_i \in E$, and $\varphi^*(b_{i,p}) = p$, for all $b_{i,p} \in B;$
- $l^*(e) = \begin{cases} tick, & if\ \varphi^*(e) = \surd, \\ L(\varphi^*(e)), & otherwise \end{cases}, for\ all\ e \in E.$

Determine $\rho^* = e_1 \dots e_n$.

Example 9. Consider the time Petri net $\widetilde{\mathcal{TN}}$ depicted in Fig. 1 and its run $\sigma = t_1 \surd t_3 \surd t_2$. We construct the sequence of the following tuples (E_i, B_i, G_i, C_i), with $0 \leq i \leq 5$.

- $i = 0$. Set $E_0 = \emptyset$; $C_0 = B_0 = \{b_{0,p} \mid p \in M_0 = \{p_1, p_2\}\} = \{b_{0,p_1}, b_{0,p_2}\}$; $G_0 = \emptyset$.
- $i = 1$. Set $E_1 = E_0 \cup \{e_1\}$; $B_1 = B_0 \cup \{b_{1,p} \mid p \in t_1\bullet = \{p_3\}\} = \{b_{1,p_3}\}$; $G_1 = G_0 \cup \{(e_1, b_{1,p}) \mid b_{1,p} \in B_1\} \cup \{(b_{j,p}, e_1) \mid b_{j,p} \in C_0,\ p \in \bullet t_1 = \{p_1\}\} = G_0 \cup \{(e_1, b_{1,p_3}), (b_{0,p_1}, e_1)\}$; $C_1 = (C_0 \setminus \bullet e_1) \cup e_1\bullet = \{b_{0,p_2}, b_{1,p_3}\}$.
- $i = 2$. Set $E_2 = E_1 \cup \{e_2\}$; $B_2 = B_1 \cup \{b_{2,p} \mid b_{j,p} \in C_1\} = \{b_{2,p_2}, b_{2,p_3}\}$; $G_2 = G_1 \cup \{(e_2, b_{2,p}) \mid b_{2,p} \in B_2\} \cup \{(b_{j,p}, e_2) \mid b_{j,p} \in C_1\} = G_1 \cup \{(e_2, b_{2,p_3}), (e_2, b_{2,p_2}), (b_{1,p_3}, e_2), (b_{0,p_2}, e_2)\}$; $C_2 = (C_1 \setminus \bullet e_2) \cup e_2\bullet = \{b_{2,p_3}, b_{2,p_2}\}$.
- $i = 3$. Set $E_3 = E_2 \cup \{e_3\}$; $B_3 = B_2 \cup \{b_{3,p} \mid p \in t_3\bullet = \{p_4\}\} = \{b_{3,p_4}\}$; $G_3 = G_2 \cup$

$\{(e_3, b_{3,p}) \mid b_{3,p} \in B_3\} \cup \{(b_{j,p}, e_3) \mid b_{j,p} \in C_2,\ p \in \bullet t_3 = \{p_2\}\} = G_3 \cup \{(e_3, b_{3,p_4}), (b_{2,p_2}, e_3)\}$; $C_3 = (C_2 \setminus \bullet e_3) \cup e_3\bullet = \{b_{2,p_3}, b_{3,p_4}\}$.

- $i = 4$. Set $E_4 = E_3 \cup \{e_4\}$; $B_4 = B_3 \cup \{b_{4,p} \mid b_{j,p} \in C_3\} = \{b_{4,p_3}, b_{4,p_4}\}$; $G_4 = G_3 \cup \{(e_4, b_{4,p}) \mid b_{4,p} \in B_4\} \cup \{(b_{j,p}, e_4) \mid b_{j,p} \in C_3\} = G_3 \cup \{(e_4, b_{4,p_3}), (e_4, b_{4,p_4}), (b_{2,p_3}, e_4), (b_{3,p_4}, e_4)\}$; $C_4 = (C_3 \setminus \bullet e_4) \cup e_4\bullet = \{b_{4,p_3}, b_{4,p_4}\}$.
- $i = 5$. Set $E_5 = E_4 \cup \{e_5\}$; $B_5 = B_4 \cup \{b_{5,p} \mid p \in t_2\bullet = \{p_1, p_2\}\} = \{b_{5,p_1}, b_{5,p_2}\}$; $G_5 = G_4 \cup \{(e_5, b_{5,p}) \mid b_{5,p} \in B_5\} \cup \{(b_{j,p}, e_5) \mid b_{j,p} \in C_4,\ p \in \bullet t_2 = \{p_3, p_4\}\} = G_4 \cup \{(e_5, b_{5,p_1}), (e_5, b_{5,p_2}), (b_{4,p_3}, e_5), (b_{4,p_4}, e_5)\}$; $C_5 = (C_4 \setminus \bullet e_5) \cup e_5\bullet = \{b_{5,p_1}, b_{5,p_2}\}$.

Determine the following: $\varphi^*(e_i) = \overline{t_i}$, for all $e_i \in E_5$, $\varphi^*(b_{j,p}) = p$, for all $b_{j,p} \in B_5$, $l^*(e_i) = \begin{cases} tick, if\ \varphi^*(e_i) = \surd, \\ L(\varphi^*(e_i)), otherwise \end{cases}$, for all $e_i \in E_5$. Identify $\pi^* = (\widetilde{TN}^* = (E_5, B_5, G_5, C_5, l^*), \varphi^*)$ and $\rho^* = e_1 \ldots e_n$. Notice that $\widetilde{TN}^*$ and $\widetilde{TN}$ from Example 6 are equal up to renaming their conditions.
□

The following lemmas demonstrate important properties of the constructions from Definition 9.

Lemma 4.
a) TN^* is a casual net;
b) $\rho^* = e_1 \ldots e_n$ is a linearization of TN^*;
c) $C_i = Cut(E_i)$, for all $0 \le i \le n$.

Proof. See Appendix. □

Lemma 5. The mapping φ^* is a homomorphism from TN^* to $\mathcal{TN}$.

Proof. See Appendix. □

We are now ready to establish that FS is a surjective mapping.

Theorem 4. Given a run σ of a time Petri net $\mathcal{TN}$, there exists an admissible time process $\pi^* = (TN^*, \varphi^*)$ of $\mathcal{TN}$ and a linearization $\rho^* = e_1 \ldots e_n$ of TN^* such that $\sigma = FS(\rho^*) = \varphi^*(e_1) \ldots \varphi^*(e_n)$.

Proof. Consider the construction of π^* from Definition 9. According to Lemma 4(a) and Lemma 5, π^* is a time process of $\mathcal{TN}$. Take an arbitrary $1 \le i \le n$ such that $\varphi^*(e_i) \in T$. Then, $\mathbf{Clock}(\bullet e_i, \varphi^*(e_i)) = I_{i-1}(\varphi^*(e_i))$, by Lemma 3(c). Due to σ being a run of $\mathcal{TN}$, we obtain that $Eft(\varphi^*(e_i)) \le \mathbf{Clock}(\bullet e_i, \varphi^*(e_i)) \le Lft(\varphi^*(e_i))$. Hence, π^* is an admissible time process of $\mathcal{TN}$. Thanks to Lemma 4(b), $\rho^* = e_1 \ldots e_n$ is a linearization of TN^*. By the construction of φ^*, we get that $FS(\rho^*) = \sigma$.
□

The following theorem shows that FS is an injection, i.e. the constructed in Definition 9 time process $\pi^* = (TN^*, \varphi^*)$ is unique up to isomorphism.

Theorem 5. Let σ be a run of $\mathcal{TN}$. The time process $\pi = (TN, \varphi)$ of $\mathcal{TN}$ with linearization $\rho = e_1 \ldots e_n$ of TN, such that $\sigma = FS(\rho) = \varphi(e_1) \ldots \varphi(e_n)$ is unique up to isomorphism.

Proof. Take arbitrary time process $\pi' = (TN' = (B', E', G', l'), \varphi')$ of $\mathcal{TN}$ and linearization $\rho' = e'_1 \ldots e'_n$ $(n \ge 0)$ of TN' such that $FS(\rho') = \sigma$. Then, $E' = \{e'_1, \ldots, e'_n\}$, by the definition of the linearization.

Moreover, $B' = \bullet TN' \oplus e'_1\bullet \oplus \ldots \oplus e'_n\bullet$, due to TN' being an acyclic net. Set $B'_0 = \bullet TN'$ and $B'_i = B'_{i-1} \cup e_i{}'\bullet$, for $1 \le i \le n$. By Definition 7, the restriction of φ' to $\bullet TN'$ is a bijection between

$\bullet TN'$ and M_0. Then, w.l.o.g. assume $B'_0 = \{b'_{0,p} | p \in M_0\}$, with $\varphi'(b'_{0,p}) = p$. Take an arbitrary $1 \le i \le n$. Suppose $\varphi'(e'_i) \in T$. Then, due to the restriction of φ' to $e'_i\bullet$ being a bijection between $e'_i\bullet$ and $\varphi'(e'_i)\bullet$, w.l.o.g. assume $e'_i\bullet = \{b'_{i,p} | p \in \varphi'(e'_i)\bullet\}$. If $\varphi'(e'_i) = \surd$, then $e'_i\bullet$ is a cut, i.e. $e'_i\bullet \ne \emptyset$, thanks to Definition 7. In addition, we have that $e'_i\bullet \subseteq Cut(E'_i)$, by Lemma 2(a), and $Cut(E'_i)$ is cut, by Lemma 2(c), i.e. $e_i{}'\bullet = Cut(E'_i)$. Thanks to Definition 2, it holds that $M_i = M_{i-1}$. Hence, the restriction of φ' to $e_i{}'\bullet$ is a bijection between $e'_i\bullet$ and $\varphi'(Cut(E'_{i-1}))$, according to Lemma 3(a). W.l.o.g. suppose $e'_i\bullet = \{b'_{i,p} \mid p \in \varphi'(Cut(E'_{i-1}))\}$. Thus, for all $1 \le i \le n$, we obtain the following:

- $B'_i = B'_{i-1} \cup \begin{cases} e'_i\bullet = \{b'_{i,p} \mid p \in \varphi'(Cut(E'_{i-1}))\}, & if\ \varphi'(e'_i) = \surd, \\ e'_i\bullet = \{b'_{i,p} \mid p \in \varphi'(e'_i)\bullet\}, & otherwise \end{cases}$;
- $\varphi'(b'_{i,p}) = p$, for all $b'_{i,p} \in B'_i$.

Compare the time process π' of $\mathcal{TN}$ and the time process $\pi^* = ((B, E, G, l^*), \varphi^*)$ of $\mathcal{TN}$ of TN^* (from Definition 9). Clearly, E' and E have the same cardinality. Due to $FS(\rho') = FS(\rho^*)$, we obtain that $\varphi'(e'_i) = \varphi^*(e_i)$, for all $1 \le i \le n$. According Lemma 3(a) and Lemma 4(c), it holds that $\varphi'(Cut(E_{i-1})) = M_{i-1} = \varphi^*(Cut(E_{i-1}) = C_{i-1})$, for all $1 \le i \le n$. Hence, B_i and B'_i have the same cardinality, for all $0 \le i \le n$.
Thanks to the definitions of E' (E^*) and B' (B^*), we can construct a bijective mapping $\gamma : (E' \cup B') \to (E \cup B)$, with $\gamma(e'_i) = e_i$, for all $e'_i \in E'$, and $\gamma(b'_{j,p}) = b_{j,p}$, for all $b'_{j,p} \in B'$ such that $\gamma(B') = B$ and $\gamma(E') = E$. Clearly, $\varphi'(x) = \varphi^*(\gamma(x))$, for all $x \in B' \cup E'$, and hence, $l'(e'_i) = l^*(\gamma(e'_i))$, for all $e'_i \in E'$. It remains to show that G' is isomorphic to G. Take an arbitrary $1 \le i \le n$. Due to the definitions of B_i and B'_i, we have that $(e'_i, b'_{j,p}) \in G' \Leftrightarrow b'_{j,p} \in B'_i \Leftrightarrow b_{j,p} \in B_i \Leftrightarrow (e_i, b_{j,p}) \in G$. Check that $(b'_{j,p}, e'_i) \in G' \Leftrightarrow (b_{j,p}, e_i) \in G$.

Claim. $b_{j,p} \in Cut(E_{i-1}) \Leftrightarrow b'_{j,p} \in Cut(E'_{i-1})$.

Proof. We prove the case with $b_{j,p} \in Cut(E_{i-1})$ (the case with $b'_{j,p} \in Cut(E'_{i-1})$ is symmetric). Then, there exists $b'_{j',p} \in Cut(E'_{i-1})$, according to Lemma 3(a). Suppose a contrary, i.e. $j' \ne j$. W.l.o.g. assume $j < j' < i$. As $\gamma : B' \to B$ is bijection, there exists $b_{j',p} = \gamma(b'_{j',p})$. Due to Definition 9, we have that $b_{j,p} \in \bullet TN$, if $j = 0$, or $b_{j,p} \in e_j\bullet \subseteq E_j\bullet$, if $j > 0$, and $b_{j',p} \in e_{j'}\bullet$, because $j' > j$. Then, $b_{j',p} \in Cut(E_{j'})$, by Lemma 2(a). However, $b_{j,p} \notin Cut(E_{j'})$, thanks to Lemma 3(a). Since $E_j\bullet \subseteq E_{j'}\bullet$, we get that $b_{j,p} \in \bullet TN \cup E_{j'}\bullet$. Hence, $b_{j,p} \in \bullet E_{j'}$, due to the definition of $Cut(E_{j'})$. This implies that $b_{j,p} \in \bullet E_{i-1}$, contradicting $b_{j,p} \in Cut(E_{i-1})$.
□

According to Lemma 2(b), we have $\bullet e'_i \subseteq Cut(E'_{i-1})$. Assume that $\varphi'(e'_i) = \surd$. Then, $\bullet e'_i$ is a cut, i.e $\bullet e'_i \ne \emptyset$, due to Definition 7. Moreover, we have that $Cut(E'_{i-1})$ is cut, by Lemma 2(c), i.e. $\bullet e'_i = Cut(E'_{i-1})$. Therefore, $(b'_{j,p}, e'_i) \in G' \Leftrightarrow b'_{j,p} \in Cut(E'_{i-1}) \Leftrightarrow b_{j,p} \in Cut(E_{i-1}) \Leftrightarrow (b_{j,p}, e_i) \in G$, thanks to Claim. Assume $\varphi'(e'_i) \in T$. Then, it holds that $\bullet e'_i = \{b'_{j,p} \mid p \in \bullet\varphi'(e'_i) \wedge b'_{j,p} \in Cut(E'_{i-1})\}$, due to the restriction of φ' to $\bullet e'_i$ being a bijection between $\bullet e'_i$ and $\bullet\varphi'(e'_i)$. By virtue of Claim, we get that $(b'_{j,p}, e'_i) \in G' \Leftrightarrow p \in \bullet\varphi'(e'_i) \wedge b'_{j,p} \in Cut(E'_{i-1}) \Leftrightarrow p \in \bullet\varphi^*(e_i) \wedge b_{j,p} \in Cut(E_{i-1}) \Leftrightarrow (b_{j,p}, e_i) \in G$.
Therefore, we obtain that $\gamma: \pi' \simeq \pi^*$.
□

Thus, we have demonstrated that FS is a bijective mapping between linearizations of time processes and runs from the initial state, in the context of the TPN $\mathcal{TN}$.

6. Conclusion

In this paper, we have introduced and studied partial order semantics for TPNs with weak time elapsing and intermediate memory policies. First, we have developed a state space discretization technique for the TPN, i.e. we have shown that any of its run with real-value time elapsings can be represented as that with the same untimed part and with only unit time elapsings. This allows us to transform time elapsings into the structure of a causal net with tick-events. Second, partial order semantics of the TPN has been proposed in the terms of time causal processes which consist of causal nets and their homomorphism into the TPN. Partial order semantics is useful for taking into account the processes' timing behavior in addition to their degrees of relative concurrency. Also, in the context of the TPN, a bijective mapping has been proved to exist between interleaving runs and computations (linearizations) of time causal processes, demonstrating that the partial order semantics is correct w.r.t. the interleaving that.

As for future work, we plan to extend the results obtained to atomic memory and back in time policies. As well, we believe that partial order semantics developed here allows us to elaborate and investigate behavioral equivalences of TPNs with weak semantics, in interleaving – partial order dichotomy.

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Appendix

Proof of Lemma 1. Let β is a solution of B_ω. We shall prove, that $(M_0, I_0) \overset{[\omega_i]_\beta}{\longrightarrow} (M_{\omega_i}, [I'_{\omega_i}]_\beta)$, for all $0 \le i \le k$, by induction on i.

$i = 0$. Due to Definition 4, we get: $\omega_0 = x_0$; $M_{\omega_0} = M_0$; $\forall t \in En(M_{\omega_0}) : I'_{\omega_0}(t) = x_0$. Hence, $(M_0, I_0) \overset{[\omega_0]_\beta}{\longrightarrow} (M_{\omega_0}, [I'_{\omega_0}]_\beta)$, thanks to Definition 2.

$i > 0$. By the induction hypothesis, we have that $(M_0, I_0) \overset{[\omega_{i-1}]_\beta}{\longrightarrow} (M_{\omega_{i-1}}, [I'_{\omega_{i-1}}]_\beta)$. Due to Definition 4, it holds:

- $\omega_i = \omega_{i-1} t_i x_i$;
- $M_{\omega_i} := (M_{\omega_{i-1}} \setminus \bullet t_i) \cup t_i \bullet$ (i.e. $t_i \in En(M_{\omega_{i-1}})$);
- $Eft(t_i) \le I'_{\omega_{i-1}}(t_i) \le Lft(t_i)$ in B_ω;
- $\forall t \in En(M_{\omega_i}), I'_{\omega_i}(t) - x_i = \begin{cases} 0, & \text{if } \uparrow enabled(t, M_{\omega_{i-1}}, t_i) \\ I'_{\omega_{i-1}}(t), & \text{otherwise} \end{cases}$.

Then, $Eft(t_i) \le [I'_{\omega_{i-1}}(t)]_\beta \le Lft(t_i)$, because β is a solution of B_ω. Therefore, t_i can fire from the state $(M_{\omega_{i-1}}, [I'_{\omega_{i-1}}]_\beta)$. By Definition 2, we have that $(M_0, I_0) \overset{[\omega_{i-1}]_\beta}{\longrightarrow} (M_{\omega_{i-1}}, [I'_{\omega_{i-1}}]_\beta) \overset{t_i}{\longrightarrow} (M_{\omega_i}, [I'_{\omega_i}]_\beta - \beta(x_i)) \overset{\beta(x_i)}{\longrightarrow} (M_{\omega_i}, [I'_{\omega_i}]_\beta)$. Hence, it is true that $(M_0, I_0) \overset{[\omega]_\beta}{\longrightarrow} (M_\omega, [I'_\omega]_\beta)$. Moreover, for all $1 \le i \le k$, we have $[I'_{\omega_{i-1}}(t_i)]_\beta = I'_{i-1}(t_i)$. □

Proof of Proposition 1.

Let $\omega = x_0 t_1 x_1 \ldots t_k x_k$; β_i be the functions from Definition 5, with $0 \le i \le k$; $\beta_\omega = \beta_k$; and $\mathcal{B}_\omega$ be the set of the variable parts of the inequalities from B_ω. In order to show that the assignment β_ω is a solution of ω, we consider an important property of the mappings β_i, for all $0 \le i \le k$.

Claim. For all $g \in \mathcal{B}_\omega$ and $0 \le i \le k$, it holds that $\lceil [g]_{\beta_i} \rceil \ge \lfloor [g]_{\beta_0} \rfloor$ and $\lfloor [g]_{\beta_i} \rfloor \le \lceil [g]_{\beta_0} \rceil$.

Proof. We shall prove by induction on i.

$i = 0$. Obvious.

$i > 0$. Take an arbitrary $g \in \mathcal{B}_\omega$. By the induction hypothesis, we have that $\lceil [g]_{\beta_{i-1}} \rceil \ge \lfloor [g]_{\beta_0} \rfloor$ and $\lfloor [g]_{\beta_{i-1}} \rfloor \le \lceil [g]_{\beta_0} \rceil$. Let $\underline{\beta_i}$ be the function from Definition 5.

Assume that does not exist $h \in \mathcal{B}_\omega \ s.t. \lceil [h]_{\underline{\beta_i}} \rceil < \lfloor [h]_{\beta_0} \rfloor$. Then, $[g]_{\beta_i} = [g]_{\underline{\beta_i}}$ and $[g]_{\beta_i} \le [g]_{\beta_{i-1}}$, due to Definition 5. Then $\lceil [g]_{\beta_i} \rceil = \lceil [g]_{\underline{\beta_i}} \rceil \ge \lfloor [g]_{\beta_0} \rfloor$ and $\lfloor [g]_{\beta_i} \rfloor \le \lfloor [g]_{\beta_{i-1}} \rfloor \le \lceil [g]_{\beta_0} \rceil$.

Assume that there is exists $h \in \mathcal{B}_\omega \ s.t. \lceil [h]_{\underline{\beta_i}} \rceil < \lfloor [h]_{\beta_0} \rfloor$. Then, $[g]_{\beta_i} \ge [g]_{\beta_{i-1}}$, by Definition 5. Therefore, $\lceil [g]_{\beta_i} \rceil \ge \lceil [g]_{\beta_{i-1}} \rceil \ge \lfloor [g]_{\beta_0} \rfloor$. Suppose a contrary, i.e. $\lfloor [g]_{\beta_i} \rfloor > \lceil [g]_{\beta_0} \rceil$. Then, $[g]_{\beta_i} \ge \lfloor [g]_{\beta_i} \rfloor \ge \lceil [g]_{\beta_0} \rceil + 1$. According to Definition 5, x_{k-i} appears in g and h and, moreover, $[h]_{\beta_i} = [h]_{\beta_{i-1}} - \beta_{i-1}(x_{k-i}) + \beta_i(x_{k-i}) = [h]_{\beta_{i-1}} - \beta_{i-1}(x_{k-i}) + \lceil \beta_{i-1}(x_{k-i}) \rceil \le [h]_{\beta_{i-1}} - \beta_{i-1}(x_{k-i}) + \lfloor \beta_{i-1}(x_{k-i}) \rfloor + 1 = [h]_{\underline{\beta_i}} + 1$. As $[h]_{\underline{\beta_i}} + 1 \le \lceil [h]_{\underline{\beta_i}} \rceil + 1 \le \lfloor [h]_{\beta_0} \rfloor$, we have $[h]_{\beta_i} \le \lfloor [h]_{\beta_0} \rfloor$. Let $S : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_{\ge 0}$ be such that $S(a, b) = \sum_{j=a}^{b} x_j$, if $a < b$, and $S(a, b) = 0$, otherwise. Due to Definition 4, it holds that $h = S(m, n)$ and $g = S(m', n')$, where $0 \le m, m' \le k - i \le n, n' \le k$. By the construction of β_k, we have that $[S(a, b)]_{\beta_c} = [S(a, b)]_{\beta_0}$, if $0 \le b < k - c \le k$; and $[S(a, b)]_{\beta_c} = [S(a, b)]_{\beta_k}$, if $0 \le k - c \le a \le k$. Therefore, we obtain the following:

(1) $[S(k-i, n)]_{\beta_i} \le \lfloor [S(k-i, n)]_{\beta_0} \rfloor$, due to $[S(m, k-i-1)]_{\beta_i} = [S(m, k-i-1)]_{\beta_0}$ and $[h]_{\beta_i} \le \lceil [h]_{\underline{\beta_i}} \rceil < \lfloor [h]_{\beta_0} \rfloor$;

(2) $[S(k-i, n')]_{\beta_i} \ge \lceil [S(k-i, n')]_{\beta_0} \rceil + 1$, due to $[S(m', k-i-1)]_{\beta_i} = [S(m', k-i-1)]_{\beta_0}$ and $[g]_{\beta_i} \ge \lceil [g]_{\beta_0} \rceil + 1$;

(3) $[S(m', n)]_{\beta_{k-n-1}} = [S(m', n)]_{\beta_0}$;

(4) $[S(m, n')]_{\beta_{k-n'-1}} = [S(m, n')]_{\beta_0}$;

(5) $[S(n+1, n')]_{\beta_{k-n-1}} = [S(n+1, n')]_{\beta_i} = [S(n+1, n')]_{\beta_k}$;

(6) $[S(n'+1, n)]_{\beta_{k-n'-1}} = [S(n'+1, n)]_{\beta_i} = [S(n'+1, n)]_{\beta_k}$.

Three cases are admissible:

- $n = n'$. Then, (1) contradicts (2).
- $n < n'$. Then, it holds that $[g]_{\beta_{k-n-1}} = [S(m', n)]_{\beta_{k-n-1}} + [S(n+1, n')]_{\beta_{k-n-1}} =^{(3)} [S(m', k-i-1)]_{\beta_0} + [S(k-i, n)]_{\beta_0} + [S(n+1, n')]_{\beta_{k-n-1}} \ge^{(1),(5)} [S(m', k-i-1)]_{\beta_0} + [S(k-i, n)]_{\beta_i} + [S(n+1, n')]_{\beta_i} \ge^{(2)} \lceil [g]_{\beta_0} \rceil + 1$. This contradicts the induction hypothesis, because $k - n \le i$.

– $n > n'$. Then, it holds that $[h]_{\beta_{k-n'-1}} = [S(m,n')]_{\beta_{k-n'-1}} + [S(n'+1,n)]_{\beta_{k-n'-1}} =^{(4)} [S(m,k-i-1)]_{\beta_0} + [S(k-i,n')]_{\beta_0} + [S(n'+1,n)]_{\beta_{k-n'-1}} \le^{(2),(6)} [S(m,k-i-1)]_{\beta_0} + [S(k-i,n')]_{\beta_i} - 1 + [S(n'+1,n)]_{\beta_i} \le^{(1)} \lfloor [h]_{\beta_0} \rfloor - 1$. We get a contradiction the induction hypothesis, because $k - n' \le i$. □

By Definition 4, x_k does not appear in B_ω. Then, β_0 is a solution of B_ω, according to Definition 5. Take an arbitrary g from $\mathcal{B}_\omega$. Then, it holds that $(a \le g \le b) \in B_\omega$, where $a, b \in \mathbb{Z}$. By Claim, we have $\lfloor [g]_{\beta_0} \rfloor \le [g]_{\beta_\omega} \le \lceil [g]_{\beta_0} \rceil$. Due to β_0 being a solution, we obtain $a \le [g]_{\beta_0} \le b$ and, moreover $a \le \lfloor [g]_{\beta_0} \rfloor, \lceil [g]_{\beta_0} \rceil \le b$, because $a, b \in \mathbb{N} \cup \infty$. Thus, it holds that $a \le \lfloor [g]_{\beta_0} \rfloor \le [g]_{\beta_\omega} \le \lceil [g]_{\beta_0} \rceil \le b$. Therefore, β_ω is a solution of B_ω. □

Proof of Lemma 2.

a) Take an arbitrary $1 \le k \le n$. By definitions, we have that $Cut(E_k) = (\bullet TN \cup E_k\bullet) \setminus \bullet E_k = (\bullet TN \cup E_{k-1}\bullet \cup e_k\bullet) \setminus (\bullet E_{k-1} \cup \bullet e_k)$. As TN is an acyclic net, we obtain that $\bullet e_k \cap e_k\bullet = \emptyset$ and $\bullet E_{k-1} \cap e_k\bullet = \emptyset$. Then, it holds that $Cut(E_k) = (((\bullet TN \cup E_{k-1}\bullet) \setminus \bullet E_{k-1}) \setminus \bullet e_k) \cup e_k\bullet = (Cut(E_{k-1}) \setminus \bullet e_k) \cup e_k\bullet$.

b) Take arbitrary $1 \le k \le n$. The case with $\bullet e_k = \emptyset$ is trivial. Suppose $b \in \bullet e_k$. By definition, we have that $Cut(E_{k-1}) = (\bullet TN \cup E_{k-1}\bullet) \setminus \bullet E_{k-1}$. Due to TN being a causal net, $b \notin \bullet e_i$, for all $1 \le i < k$. Hence, $b \notin \bullet E_{k-1}$. If $b \in \bullet TN$, then $b \in Cut(E_{k-1})$. Consider the case when $b \notin \bullet TN$. Then, there is e_i such that $b \in e_i\bullet$. Clearly, $e_i \prec e_k$. This implies that $i < k$, in the linearization ρ. Hence, $b \in E_{k-1}\bullet$ and $b \in Cut(E_{k-1})$.

c) As $Cut(E_0) = \bullet TN = \{b \in B \mid \bullet b = \emptyset\}$, we have $Cut(E_0)$ is a co-set. Suppose a contrary, i.e. there are $b, b' \in Cut(E_k)$, for some $1 \le k \le n$, such that $b \prec b'$. As ρ is a linearization, we have $bGe_i \ldots e_j Gb'$, with $i \le j$. Due to $Cut(E_k) = (\bullet TN \cup E_k\bullet) \setminus \bullet E_k$, we get $b, b' \notin \bullet E_k$ and $b' \in E_k\bullet$. Since $\bullet b' = e_j$, it holds that $j \le k$, i.e. $i \le k$. This means that $b \in \bullet e_i \subseteq \bullet E_k$, contradicting $b \notin \bullet E_k$. Thus, $\neg(b \prec b')$.

We shall show that $Cut(E_k)$ is a cut, for all $0 \le k \le n$. Suppose a contrary, i.e. there exists $b \notin Cut(E_k)$, for some $0 \le k \le n$, such that $\neg(b \prec b')$ and $\neg(b' \prec b)$, for all $b' \in Cut(E_k)$. W.l.o.g. assume $b \in Cut(E_i)$, for some $0 \le i \ne k \le n$. Thanks to item a), $Cut(E_j) = (Cut(E_{j-1}) \setminus \bullet e_j) \cup e_j\bullet$, for all $1 \le j \le n$. If $i < k$, then we get that $bGe_l \ldots e_m Gb'$, for some $b' \in Cut(E_k)$ and $i < l \le m \le k$, i.e. $b \prec b'$, because TN is a causal net. If $i > k$, then we have that $b'Ge_l \ldots e_m Gb$, for some $b' \in Cut(E_k)$ and $k < l \le m \le i$, i.e. $b' \prec b$, again because TN is a causal net. Thus, $Cut(E_k)$ is a cut, for all $0 \le k \le n$.
□

Proof of Lemma 3.

a), b) We shall verify the items by induction on $0 \le i \le n$.

$i = 0$. By definitions, it holds that $Cut(E_0) = \bullet TN$.

a) The restriction of φ to $Cut(E_0)$ is a bijection between $Cut(E_0)$ and M_0, due to Definition 7.

b) As $Age(b) = 0$, for all $b \in \bullet TN$, $\mathbf{Clock}(Cut(E_0), t) = 0 = I_0(t)$, for all $t \in En(M_0)$, thanks to $\mathcal{TN}$ being T-restricted.

$i > 0$. By the induction hypothesis, the items hold for $i - 1$. We now check them for i. Two cases are admissible.

Case 1: $\varphi(e_i) = \surd$. Then, it holds that $M_{i-1} = M_i$, by Definition 2. According to Definition 7, we have that $\varphi(e_i\bullet) = \varphi(\bullet e_i)$ and $\bullet e_i, e_i\bullet$ are cuts, i.e. $\bullet e_i \ne \emptyset, e_i\bullet \ne \emptyset$. In addition, we have that $\bullet e_i \subseteq Cut(E_{i-1})$, $e_i\bullet \subseteq Cut(E_i)$, and $Cut(E_{i-1})$, $Cut(E_i)$ are cuts, due to Lemma 2. Hence, we get that $\bullet e_i = Cut(E_{i-1})$ and $Cut(E_i) = e_i\bullet$, and, moreover, $\varphi(Cut(E_i)) = \varphi(Cut(E_{i-1})) = M_{i-1} = M_i$.

a) As the restriction of φ to $e_i\bullet$ is an injection, by Definition 7, the restriction of φ to $Cut(E_i)$ is a bijection between $Cut(E_i)$ and M_i.

b) Take an arbitrary $t \in En(M_i)$. According to Definition 2, we have that $t \in En(M_{i-1})$ and $I_{i-1}(t) + \varphi(e_i) = I_i(t)$. By definition, $Age(b) = Age(b') + \varphi(e_i)$, with $b' \in Cut(E_{i-1})$ and $\varphi(b) = \varphi(b')$, for all $b \in e_i\bullet \in Cut(E_i)$. Then, due to $\mathcal{TN}$ being T-restricted, we obtain that $\mathbf{Clock}(Cut(E_i), t) = min(\{Age(b) | \varphi(b) \in \bullet t, b \in Cut(E_i)\}) = min(\{Age(b) + \varphi(e_i) | \varphi(b) \in \bullet t, b \in Cut(E_{i-1})\}) = \mathbf{Clock}(Cut(E_{i-1}), t) + \varphi(e_i) = I_{i-1}(t) + \varphi(e_i) = I_i(t)$.

Case 2: $\varphi(e_i) \in T$. Then, $M_i = (M_{i-1} \setminus \bullet\varphi(e_i)) \cup \varphi(e_i)\bullet$, according to Definition 2. Due to Definition 7, the restrictions of φ to $\bullet e_i$ ($e_i\bullet$) are bijections between $\bullet e_i$ ($e_i\bullet$) and $\bullet\varphi(e_i)$ ($\varphi(e_i)\bullet$).

a) By the inductive hypothesis, the restriction of φ to $Cut(E_{i-1})$ is a bijection between $Cut(E_{i-1})$ and M_{i-1}. Then, due to Definition 2, we get that $M_i = (\varphi(Cut(E_{i-1})) \setminus \bullet\varphi(e_i)) \cup \varphi(e_i)\bullet = \varphi((Cut(E_{i-1}) \setminus \bullet e_i) \cup e_i\bullet) = \varphi(Cut(E_i))$, using Lemma 2(a). Since $\mathcal{TN}$ is contact-free, we obtain that $\varphi(Cut(E_{i-1}) \setminus \bullet e_i) \cap \varphi(e_i\bullet) = \emptyset$. Therefore, the restriction of φ to $Cut(E_i)$ is a bijection between $Cut(E_i)$ and M_i.

b) Take an arbitrary $t \in En(M_i)$. Assume that $\uparrow enabled(t, M_{i-1}, \varphi(e_i))$ is true. Then, we have that $t \notin En(M_{i-1} \setminus \bullet\varphi(e_i))$ or $t = \varphi(e_i)$, by Definition 2. If $t = \varphi(e_i)$, then $t \in En(M_{i-1})$ and $t \notin En(M_{i-1} \setminus \bullet t) = En(M_{i-1} \setminus \bullet\varphi(e_i))$. So, $t \notin En(M_{i-1} \setminus \bullet\varphi(e_i))$. Due to Definition 2, it is true that $En(M_i) = En((M_{i-1} \setminus \bullet\varphi(e_i)) \cup \varphi(e_i\bullet))$. Thanks to $\mathcal{TN}$ is T-restricted, we get $\bullet t \ne \emptyset$ and $\varphi(e_i\bullet) \ne \emptyset$. Since $t \in En(M_i)$ and $t \notin En(M_{i-1} \setminus \bullet\varphi(e_i))$, we have that $\bullet t \cap \varphi(e_i\bullet) \ne \emptyset$. According to Lemma 2(a), it holds that $e_i\bullet \subseteq Cut(E_i)$. Hence, there is $b \in Cut(E_i)$ such that $\varphi(b) \in \bullet t$ and $Age(b) = 0$. Therefore, due to $\mathcal{TN}$ being T-restricted, it is true that $\mathbf{Clock}(Cut(E_i), t) = (\min\{Age(b) | \varphi(b) \in \bullet t, b \in Cut(E_i)\}) = 0$.
Thus, $\mathbf{Clock}(Cut(E_i), t) = I_i(t)$, due to Definition 2.
Suppose that $\uparrow enabled(t, M_{i-1}, \varphi(e_i))$ is false. Then, we get that $t \in En(M_{i-1} \setminus \bullet\varphi(e_i))$ and $t \ne \varphi(e_i)$, by Definition 2. Hence, $\bullet\varphi(e_i) \cap \bullet t = \emptyset$, i.e. $\varphi(\bullet e_i) \cap \bullet t = \emptyset$. As $\mathcal{TN}$ is contact-free, it holds that $(M_{i-1} \setminus \bullet\varphi(e_i)) \cap \varphi(e_i)\bullet = \emptyset$. This means that, $\varphi(e_i)\bullet \cap \bullet t = \emptyset$, i.e. $\varphi(e_i\bullet) \cap \bullet t = \emptyset$,. Therefore, if $\varphi(b) \in \bullet t$, then $b \notin e_i\bullet$ and $b \notin \bullet e_i$. By the induction hypothesis, we have that $I_{i-1}(t) = \mathbf{Clock}(Cut(E_{i-1}), t) = min(\{Age(b) | \varphi(b) \in \bullet t, b \in Cut(E_{i-1})\}) = min(\{Age(b) | \varphi(b) \in \bullet t, b \in (Cut(E_{i-1}) \setminus \bullet e_i) \cup e_i\bullet\})$. Thanks to Lemma 2(a), we obtain that $I_{i-1}(t) = \mathbf{Clock}(Cut(E_i), t)$. Thus, it holds that $I_i(t) = I_{i-1}(t) = \mathbf{Clock}(Cut(E_i), t)$, due to Definition 2.

c) Assume that $i < n$ and $\varphi(e_{i+1}) \in En(M_i)$. Then, $\varphi(e_{i+1}) \in En(\varphi(Cut(E_i)))$, due to item a). By definition, due to $\mathcal{TN}$ being T-restricted, we have that $\mathbf{Clock}(Cut(E_i), \varphi(e_{i+1})) = min(\{Age(b) | \varphi(b) \in \bullet\varphi(e_{i+1}), b \in Cut(E_i)\})$. Take an arbitrary $b \in Cut(E_i)$ such that $\varphi(b) \in \bullet\varphi(e_{i+1})$. Thanks to the definition of a homomorphism, it holds that $\varphi(b) \in \varphi(\bullet e_{i+1})$. Hence, $b \in \bullet e_{i+1}$, due to item a). By virtue of Lemma 2(b), $\bullet e_{i+1} \subseteq Cut(E_i)$. This implies that $\mathbf{Clock}(Cut(E_i), \varphi(e_{i+1})) = \mathbf{Clock}(\bullet e_{i+1}, \varphi(e_{i+1}))$. Therefore, $\mathbf{Clock}(\bullet e_{i+1}, \varphi(e_{i+1})) = I_i(\varphi(e_{i+1}))$, due to item (b).
□

Proof of Lemma 4.

a) By the construction of TN^*, we have the following. First, B and E are finite sets. Second, $G \subseteq (B \times E) \cup (E \times B)$ is a flow relation such that $e_j G_i b_{j,p} G_i e_i$, i.e. $j < i$. Hence, TN^* is acyclic. Third, $l^*: E \to Act \cup \{tick\}$ is a labeling function. By the construction of G, we obtain that $\bullet b_{0,p} = \emptyset$ and $\bullet b_{i,p} = \{e_i\}$, for all $1 \le i \le n$. Therefore, $|\bullet b_{i,p}| \le 1$, for all $0 \le i \le n$. Suppose a contrary, i.e. $|b_{j,p}\bullet| > 1$, for some $b_{j,p} \in B$. Then, there exists $i \ne i'$ such that $\{b_{j,p}\} \in \bullet e_i$ and $\{b_{j,p}\} \in \bullet e_{i'}$. Hence, by the construction of G, we get that $j < i$, $b_{j,p} \in C_{i-1}$ and $j < i'$, $b_{j,p} \in C_{i'-1}$. W.l.o.g. assume $i < i'$. As $C_l = (C_{l-1} \backslash \bullet e_l) \cup e_l \bullet$, for all $1 \le l \le n$, there exists $i \le k \le i' - 1$ such that $b_{j,p} \in e_k \bullet$. According to the construction of G_k, $j = k$, contradicting $j < k$.

b) Due to Definition 9, every event of TN^* appears in the sequence $\rho^* = e_1 \dots e_n$ exactly once. By the construction of G, it holds that $e_i \prec e_j$ implies $i < j$.

c) As $C_0 = \bullet TN^*$, we get $C_0 = Cut(E_0)$. Thanks to items a), b) and Lemma 2(a), we obtain $C_i = Cut(E_i)$, for $0 \le i \le n$.

□

Proof of Lemma 5.

Claim. The restriction of φ^* to C_i is a bijection between C_i and M_i, for all $0 \le i \le n$.

Proof. We prove by induction on $0 \le i \le n$.

$i = 0$. Then, $C_0 = B_0 = \{b_{0,p} | \ p \in M_0\}$, i.e. the restriction of φ^* to C_0 is a bijection between C_0 and M_0.

$i > 0$. By the induction hypothesis, the restriction of φ^* to C_{i-1} is a bijection between C_{i-1} and M_{i-1}. Two cases are admissible.

Let $\varphi^*(e_i) = \overline{t_i} = \surd$. By Definition 2, we have $M_i = M_{i-1}$. Thanks to the construction of G_i and C_i, it holds that $C_i = (C_{i-1} \backslash \bullet e_i) \cup e_i \bullet = e_i \bullet = C_{i-1}$. Then, the restriction of φ^* to C_i is a bijection between C_i and M_i.

Let $\varphi^*(e_i) = \overline{t_i} \in T$. By Definition 2, we have that $M_i = (M_{i-1} \backslash \bullet\varphi^*(e_i)) \cup \varphi^*(e_i)\bullet$. Take an arbitrary $p \in \bullet\varphi^*(e_i)$ (it exists because $\mathcal{TN}$ is T-restricted). Then, $p \in M_{i-1}$ and, moreover, there exists $b_{j,p} \in C_{i-1}$, due to the induction hypothesis. Thanks to the construction of G_i, we get that $\bullet e_i = \{b_{j,p} | \ p \in \bullet\varphi^*(e_i) \wedge b_{j,p} \in C_{i-1}\}$ and $e_i \bullet = \{b_{i,p} | \ p \in \varphi^*(e_i)\bullet\}$. Hence, the restriction of φ^* to $\bullet e_i \ (e_i\bullet)$ is a bijection between $\bullet e_i \ (e_i\bullet)$ and $\bullet\varphi^*(e_i) \ (\varphi^*(e_i)\bullet)$. Then, $\varphi^*(C_i) = \varphi^*((C_{i-1} \backslash \bullet e_i) \cup e_i\bullet) = (\varphi^*(C_{i-1}) \backslash \varphi^*(\bullet e_i)) \cup \varphi^*(e_i\bullet) = (M_{i-1} \backslash \bullet\varphi^*(e_i)) \cup \varphi^*(e_i)\bullet = M_i$. Due to $\mathcal{TN}$ being contact-free, we obtain $\left(M_{i-1} \backslash \bullet\varphi^*(e_i)\right) \cup \varphi^*(e_i)\bullet = \emptyset$.

Therefore, the restriction of φ^* to C_i is a bijection between C_i and M_i. □

By definition, we have that $\varphi^*(B) \subseteq P, \varphi^*(E) \subseteq (T \cup \{\surd\})$, and $l^*(e) = \begin{cases} tick, if \, \varphi^*(e) = \surd, \\ L(\varphi^*(e)), otherwise \end{cases}$, for all $e \in E$.

Take an arbitrary $1 \le i \le n$.

Assume $\varphi^*(e_i) \in T$. Due to Claim, the restriction of $\varphi^*$to $\bullet e_i \ (e_i\bullet)$ is a bijection between $\bullet e_i \ (e_i\bullet)$ and $\bullet\varphi^*(e_i) \ (\varphi^*(e_i)\bullet)$.

Assume $\varphi^*(e_i) = \surd$. Thanks to the construction of G_i and C_i, we have that $C_{i-1} = \bullet e_i = e_i\bullet = C_i$. By Claim, the restriction of φ to $\bullet e_i$ and the restriction of φ to $e_i\bullet$ are injections. As $\mathcal{TN}$ is T-restricted, we obtain $\bullet e_i \ne \emptyset$ and $e_i\bullet \ne \emptyset$. According to Lemma 4(c), we have that $\bullet e_i = Cut(E_{i-1})$ and $e_i\bullet = Cut(E_i)$. Due to Lemma 2(c), $\bullet e_i$ and $e_i\bullet$ are cuts.

As $\bullet TN^* = C_0$, by construction, the restriction of φ^* to $\bullet TN^*$ is a bijection between $\bullet TN^*$ and M_0, due to Claim.

Thus, φ^* is a homomorphism from TN^* to $\mathcal{TN}$, by virtue of Lemma 4(a).

□

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